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Plücker-type relations for orthogonal planes

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Abstract

We explore a Plücker-type relation which occurs naturally in the study of maximally supersymmetric solutions of certain supergravity theories. This relation generalises at the same time the classical Plücker relation and the Jacobi identity for a metric Lie algebra and coincides with the Jacobi identity of a metric n-Lie algebra. In low dimension we present evidence for a geometric characterisation of the relation in terms of middle-dimensional orthogonal planes in Euclidean or Lorentzian inner product spaces.

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1. Introduction and main result

The purpose of this note is to present a conjectural Plücker-style formula for middle-dimensional orthogonal planes in real vector spaces equipped with an inner product of Euclidean or Lorentzian signatures. The formula is both a natural generalisation of the classical Plücker formula and of the Jacobi identity for Lie algebras admitting an invariant scalar product. The formula occurs naturally in the study of maximally supersymmetric solutions of 10-dimensional type IIB supergravity and also in six-dimensional chiral supergravity. We will state the conjecture and then prove it for special cases which have found applications in physics. To place it in its proper mathematical context we start by reviewing the classical Plücker relations. For a recent discussion, see the paper [1] by Eastwood and Michor.

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1.1. The classical Plücker relations

The classical Plücker relations describe the projective embedding of the Grassmannian of planes. Let $\mathbb{V}$ be a d-dimensional vector space (over $\mathbb{R}$ or $\mathbb{C}$, say) and let $\mathbb{V}^*$ be the dual. Let $\Lambda^p\mathbb{V}^*$ denote the space of p-forms on $\mathbb{V}$ and $\Lambda^p\mathbb{V}$ the space of p-polyvectors on $\mathbb{V}$. We shall say that a p-form F is *simple* (or *decomposable*) if it can be written as the wedge product of p 1-forms. Every (non-zero) simple p-form defines a p-plane $\Pi \subset \mathbb{V}^*$, by declaring Π to be the span of the p 1-forms. Conversely, to such a p-plane Π one can associate a simple p-form by taking a basis and wedging the elements together. A different choice of basis merely results in a non-zero multiple (the determinant of the change of basis) of the simple p-form. This means that the space of p-planes is naturally identified with the subset of the projective space of the p-forms corresponding to the rays of simple p-forms. The classical Plücker relations (see, e.g., [1,2]) give the explicit embedding in terms of the intersection of a number of quadrics in $\Lambda^p\mathbb{V}^*$. Explicitly one has the following theorem.

Theorem 1. *A p-form $F \in \Lambda^p\mathbb{V}^*$ is simple if and only if for every $(p-1)$-polyvector $\Xi \in \Lambda^{p-1}\mathbb{V}$,*

$$\iota_\Xi F \wedge F = 0,$$

where $\iota_\Xi F$ denotes the 1-form obtained by contracting F with Ξ.

Being homogeneous, these equations are well defined in the projective space $\mathbb{P}\Lambda^p\mathbb{V}^* \cong \mathbb{P}^{\binom{d}{p}-1}$, and hence define an algebraic embedding there of the Grassmannian $\mathrm{Gr}(p,d)$ of p-planes in d dimensions.

The Plücker relations arise naturally in the study of maximally supersymmetric solutions of 11-dimensional supergravity [3,4]. Indeed, the Plücker relations for the 4-form F_4 in 11-dimensional supergravity arise from the zero curvature condition for the supercovariant derivative. A similar analysis for 10-dimensional type IIB supergravity [4] yields new (at least to us) Plücker-type relations, to which we now turn.

1.2. Orthogonal Plücker-type relations

Let $\mathbb{V}$ be a real vector space of finite dimension equipped with a Euclidean or Lorentzian inner product $\langle -,-\rangle$. Let $F \in \Lambda^p\mathbb{V}^*$ be a p-form and let $\Xi \in \Lambda^{p-2}\mathbb{V}$ be a $(p-2)$-polyvector. The contraction $\iota_\Xi F$ of F with Ξ is a 2-form on $\mathbb{V}$ and hence gives rise to an element of the Lie algebra $\mathfrak{so}(\mathbb{V})$. If $\omega \in \Lambda^2\mathbb{V}^* \cong \mathfrak{so}(\mathbb{V})$, we will denote its action on a form $\Omega \in \Lambda\mathbb{V}^*$ by $[\omega,\Omega]$. Explicitly, if $\omega = \alpha\wedge\beta$, for $\alpha,\beta \in \mathbb{V}^*$, then

$$[\alpha\wedge\beta,\Omega] = \alpha\wedge\iota_{\beta^\sharp}\Omega - \beta\wedge\iota_{\alpha^\sharp}\Omega,$$

where $\alpha^\sharp \in \mathbb{V}$ is the dual vector to α defined using the inner product. We then extend linearly to any 2-form ω.

Let F_1 and F_2 be two simple forms in $\Lambda^p\mathbb{V}^*$. For the purposes of this note we will say that F_1 and F_2 are *orthogonal* if the d-planes $\Pi_i \subset \mathbb{V}$ that they define are orthogonal;

that is, $\langle X_1, X_2\rangle = 0$ for all $X_i \in \Pi_i$. Note that if the inner product in $\mathbb{V}$ is of Lorentzian signature then orthogonality does not imply that $\Pi_1 \cap \Pi_2 = 0$, as they could have a null direction in common. If this is the case, $F_i = \alpha \wedge \Theta_i$, where α is a null form and Θ_i are orthogonal simple forms in a Euclidean space in two dimensions less. Far from being a pathology, the case of null forms plays an important role in the results of Figueroa-O'Farrill and Papadopoulos [4] and is responsible for the existence of a maximally supersymmetric plane wave in IIB supergravity [5].

We now can state the following conjecture.

Conjecture 1.

(i) *Let $p \geq 2$ and $F \in \Lambda^p\mathbb{V}^*$ be a p-form on a d-dimensional Euclidean or Lorentzian inner product space $\mathbb{V}$, where $d = 2p$ or $d = 2p + 1$. For all $(p - 2)$-polyvectors $\Xi \in \Lambda^{p-2}\mathbb{V}$, the equation*

$$[\iota_\Xi F, F] = 0 \tag{1}$$

is satisfied if and only if F can be written as a sum of two orthogonal simple forms; that is,

$$F = F_1 + F_2$$

where F_1 and F_2 are simple and $F_1 \perp F_2$.

(ii) *Let $p \geq 2$ and $F \in \Lambda^p\mathbb{V}^*$ be a p-form on the Euclidean or Lorentzian vector space $\mathbb{V}$ with dimension $p \leq d < 2p$. Eq. (1) holds if and only if F is simple.*

Again the equation is homogeneous, hence its zero locus is well defined in the projective space of $\mathbb{P}\Lambda^p\mathbb{V}^* \cong \mathbb{P}^{\binom{d}{p}-1}$.

Relative to a basis $\{e_i\}$ for $\mathbb{V}$ relative to which the inner product has matrix g_{ij}, we can rewrite Eq. (1) as

$$\sum_{k,\ell=1}^{d} g^{k\ell} F_{ki_1i_2\cdots i_{p-2}[j_1} F_{j_2j_3\cdots j_p]\ell} = 0,$$

which shows that the "if" part of the conjecture follows trivially: simply complete to a pseudo-orthonormal basis for $\mathbb{V}$ the bases for the planes Π_i, express this equation relative to that basis and observe that every term vanishes.

Finally let us remark as a trivial check that both Eq. (1) and the conclusion of the conjecture are invariant under the orthogonal group $\mathrm{O}(\mathbb{V})$. A knowledge of the orbit decomposition of the space of p-forms in $\mathbb{V}$ under $\mathrm{O}(\mathbb{V})$ might provide some further insight into this problem.

To this date the first part of the conjecture has been verified for the following cases:

- $p \leq 2$: both for Euclidean and Lorentzian signatures,
- $d = 6$, $p = 3$: both for Euclidean and Lorentzian signatures,
- $d = 7$, $p = 3$: for Euclidean signature,

- $d = 8$, $p = 4$: for Euclidean signature, and
- $d = 10$, $p = 5$: for Euclidean and Lorentzian signatures.

It is the latter case which is required in the investigation of maximally supersymmetric solutions of 10-dimensional type IIB supergravity [4], whereas the second case enters in the case of six-dimensional $(1, 0)$ supergravity [6]. The fourth is expected to have applications in eight-dimensional supergravity theories.

The second part of the conjecture has been verified in the cases:

- $p \leq 2$: both for Euclidean and Lorentzian signatures,
- $d < 6$, $p = 3$: both for Euclidean and Lorentzian signatures,
- $d < 8$, $p = 4$: for Euclidean signature.

There are two conditions in the hypothesis which seem artificial at first:

- the restriction on the signature of the inner product, and
- the restriction on the dimension of the vector space.

These conditions arise from explicit counterexamples for low p, which we now discuss together with a Lie algebraic re-interpretation of the identity (1).

Before we proceed to explain these, let us remark that it might just be the case that the restriction on the dimension of the vector space is an artefact of low p. We have no direct evidence of this, except for the following. We depart from the observation that the ratio of the number of relations to the number of components of a p-form in d dimensions is $\binom{d}{p-2}$. For fixed p and large d, this ratio behaves as d^{p-2}. So for $p = 2$, the ratio is 1 and for $p = 3$ grows linearly as d. It is the latter case where the counterexamples that justify the restriction on the dimensions will be found. For $p > 3$ this ratio grows much faster and it is perhaps not unreasonable to expect that the only solutions are those which verify the conjecture.

1.3. *The case $p = 2$*

Let us observe that for $p = 2$ there are no equations, since $[F, F] = 0$ trivially in $\mathfrak{so}(\mathbb{V})$. The conjecture would say that any $F \in \mathfrak{so}(\mathbb{V})$ can be "skew-diagonalised". In Euclidean signature this is true: it is the conjugacy theorem for Cartan subalgebras of $\mathfrak{so}(\mathbb{V}) \cong \mathfrak{so}(d)$. The result also holds in Lorentzian signature; although it is more complicated, since depending on the type of element (elliptic, parabolic or hyperbolic) of $\mathfrak{so}(\mathbb{V}) \cong \mathfrak{so}(1, d-1)$, it conjugates to one of a set of normal forms, all of which satisfy the conjecture.

The conjecture does not hold in signature $(2, d)$ for any $d \geq 2$, as a quick glance at the normal forms of elements of $\mathfrak{so}(2, d)$ under $\mathrm{O}(2, d)$ shows that there are irreducible blocks of dimension higher than 2. In other words, there are elements $\omega \in \mathfrak{so}(2, d)$ for which there is no decomposition of $\mathbb{R}^{2,d}$ into 2-planes stabilised by ω. A similar situation holds in signature (p, q) for $p, q > 2$, as can be gleaned from the normal forms tabulated in [7].

This justifies restricting the signature of the scalar product on $\mathbb{V}$ in the hypothesis to the conjecture. The restriction on the dimension of $\mathbb{V}$ arises by studying the case $p = 3$, to which we now turn.

1.4. The case $p = 3$

Let $F \in \Lambda^3\mathbb{V}^*$. Using the scalar product F defines a linear map $[-,-] : \Lambda^2\mathbb{V} \to \mathbb{V}$ by

$$F(X, Y, Z) = \langle [X, Y], Z\rangle, \quad \text{for all} \quad X, Y, Z \in \mathbb{V}. \tag{2}$$

The Plücker formula (1) in this case is nothing but the statement that for all $X \in \mathbb{V}$, the map $Y \mapsto [X, Y]$ should be a derivation over $[-,-]$:

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]. \tag{3}$$

In other words, it is the Jacobi identity for $[-,-]$, turning $\mathbb{V}$ into a Lie algebra, as the notation already suggests. More is true, however, and because of the fact that $F \in \Lambda^3\mathbb{V}^*$, the metric is invariant:

$$\langle [X, Y], Z\rangle = \langle X, [Y, Z]\rangle.$$

In other words, solutions of (1) for $p = 3$ are in one-to-one correspondence with Lie algebras admitting an invariant non-degenerate scalar product.

We will show below (in two different ways) that the conjecture works for $d \leq 7$, but the simple Lie algebra $\mathfrak{su}(3)$ with the Killing form provides a counterexample to the conjecture for $d = 8$ (and also for any $d > 8$ by adding to it an Abelian factor). To see this, suppose that the 3-form F associated to $\mathfrak{su}(3)$ decomposed into a sum[1] $F = F_1 + F_2$ of orthogonal simple forms. Each F_i defines a three-plane in $\mathfrak{su}(3)$. Let $Z \in \mathfrak{su}(3)$ be orthogonal to both of these planes: such Z exists because $\dim \mathfrak{su}(3) = 8$. Then $\iota_Z F = 0$, and this would mean that for all X, Y, $F(Z, X, Y) = \langle [Z, X], Y\rangle = 0$, so that Z is central, which is a contradiction because $\mathfrak{su}(3)$ is simple.

1.5. Metric n-Lie algebras

There is another interpretation of the Plücker relation (1) in terms of a generalisation of the notion of Lie algebra.[2]

Let $p = n + 1$ and $F \in \Lambda^{n+1}\mathbb{V}^*$ and as we did for $p = 3$ let us define a map $[\cdots] : \Lambda^n\mathbb{V} \to \mathbb{V}$ by

$$F(X_1, X_2, \ldots, X_{n+1}) = \langle [X_1, \ldots, X_n], X_{n+1}\rangle. \tag{4}$$

The relation (1) now says that for all $X_1, \ldots, X_{n-1} \in \mathbb{V}$, the endomorphism of $\mathbb{V}$ defined by $Y \mapsto [X_1, \ldots, X_{n-1}, Y]$ is a derivation over $[\cdots]$; that is,

$$[X_1, \ldots, X_{n-1}, [Y_1, \ldots, Y_n]] = \sum_{i=1}^{n} [Y_1, \ldots, [X_1, \ldots, X_{n-1}, Y_i], \ldots, Y_n]. \tag{5}$$

[1] For general p and d, there is no reason why F should break up as $F = F_1 + F_2$; in the general case we would have $F = F_1 + F_2 + \cdots$, where the F_i are simple and mutually orthogonal.

[2] We are grateful to Dmitriy Rumynin for making us aware of the existence of this concept.

Eq. (5) turns $\mathbb{V}$ into an *n-Lie algebra*, a notion introduced in [8] and studied since by many authors.[3] (Notice that, perhaps unfortunately, in this notation, a Lie algebra is a 2-Lie algebra.) More is true, however, and again the fact that $F \in \Lambda^{n+1}\mathbb{V}^*$ means that

$$\langle [X_1, \ldots, X_{n-1}, X_n], X_{n+1}\rangle = -\langle [X_1, \ldots, X_{n-1}, X_{n+1}], X_n\rangle, \tag{6}$$

which we tentatively call an n-Lie algebra with an invariant metric, or a *metric n-Lie algebra* for short.

To see that Eqs. (1) and (5) are the same, let us first rewrite Eq. (1) as follows:

$$\sum_a \iota_{\boldsymbol{X}} F^a \wedge F_a = 0,$$

where $\boldsymbol{X}$ stands for a $(n-1)$-vector $X_1 \wedge \cdots \wedge X_{n-1}$, and where $F_a = \iota_{e_a} F$ and $F^a = \iota_{\mathrm{e}^a} F$ with $e_a = g_{ab}\,\mathrm{e}^b$. Contracting the above equation with $n+1$ vectors $Y_1, \ldots, Y_{n+1}$, we obtain

$$\sum_a (\iota_{\boldsymbol{X}} F^a \wedge F_a)(Y_1, Y_2, \ldots, Y_{n+1}) = 0,$$

which can be rewritten as

$$\sum_{i=1}^{n+1} (-1)^{i-1} \langle [X_1, \ldots, X_{n-1}, Y_i], [Y_1, \ldots, \hat{Y}_i, \ldots, Y_{n+1}]\rangle = 0,$$

where a hat over a symbol denotes its omission. This equation is equivalent to

$$\langle [X_1, \ldots, X_{n-1}, Y_{n+1}], [Y_1, \ldots, Y_n]\rangle = \sum_{i=1}^{n} (-1)^{n-i} \langle [X_1, \ldots, X_{n-1}, Y_i], [Y_1, \ldots, \hat{Y}_i, \ldots, Y_{n+1}]\rangle.$$

Finally we use the invariance property (6) of the metric to arrive at

$$\langle [X_1, \ldots, X_{n-1}, [Y_1, \ldots, Y_n]], Y_{n+1}\rangle = \sum_{i=1}^{n} \langle [Y_1, \ldots, [X_1, \ldots, X_{n-1}, Y_i], \ldots, Y_n], Y_{n+1}\rangle,$$

which, since this is true in particular for all Y_{n+1}, agrees with (4).

There seems to be some structure theory for n-Lie algebras but to our knowledge so far nothing on metric n-Lie algebras. Developing this theory further one could perhaps gain further insight into this conjecture. We are not aware of a notion of n-Lie group, but if it did exist then both $\mathrm{Ad\,S}_5 \times S^5$ and the IIB Hpp-wave would be examples of 4-Lie groups!

[3] This structure is sometimes also called a *Filippov* algebra.

2. Verifications in low dimension

To verify the conjecture in the cases mentioned above, we shall use some group theory and the fact that any 2-form can be skew-diagonalised by an orthogonal transformation, to write down an ansatz for the p-form which we then proceed to analyse systematically. Some of the calculations leading to the verification of the conjecture have been done or checked with *Mathematica* and are contained in notebooks which are available upon request. Since the inner product allows us to identify $\mathbb{V}$ and its dual $\mathbb{V}^*$, we will ignore the distinction in what follows.

2.1. Proof for $F \in \Lambda^3\mathbb{E}^6$

Let $F \in \Lambda^3\mathbb{E}^6$ be a 3-form in six-dimensional Euclidean space. There is an orthonormal basis $\{e_1, e_2, \dots, e_6\}$ for which the 2-form $\iota_1 F$ obtained by contracting e_1 into F takes the form

$$\iota_1 F = \alpha e_{23} + \beta e_{45},$$

where $e_{ij} = e_i \wedge e_j$ and similarly for $e_{ij\cdots k}$ in what follows.

We must distinguish several cases depending on whether α and β are generic or not. In the general case, $\iota_1 F$ is a generic element of a Cartan subalgebra of $\mathfrak{so}(4)$ acting on $\mathbb{E}^4 = \mathbb{R}\langle e_2, e_3, e_4, e_5\rangle$. The non-generic cases are in one-to-one correspondence with conjugacy classes of subalgebras of $\mathfrak{so}(4)$ of strictly lower rank. In summary we have the following cases to consider:

(1) $\mathfrak{so}(4)$: α and β generic,
(2) $\mathfrak{su}(2)$: $\alpha = \pm\beta \neq 0$, and
(3) $\mathfrak{so}(2)$: $\beta = 0$, $\alpha \neq 0$.

We now treat each case in turn.

2.1.1. $\mathfrak{so}(4)$

In the first case, α and β are generic, whence the equation $[\iota_1 F, F] = 0$ says that only terms invariant under the maximal torus generated by $\iota_1 F$ survive, whence

$$F = \alpha e_{123} + \beta e_{145} + \gamma e_{236} + \delta e_{456}.$$

The remaining equations $[\iota_i F, F] = 0$ are satisfied if and only if

$$\alpha\beta + \gamma\delta = 0. \tag{7}$$

Therefore we see that indeed

$$F = (\alpha e_1 + \gamma e_6) \wedge e_{23} + (\beta e_1 + \delta e_6) \wedge e_{45}$$

can be written as the sum of two simple forms which moreover are orthogonal, since Eq. (7) implies that

$$(\alpha e_1 + \gamma e_6) \perp (\beta e_1 + \delta e_6).$$

2.1.2. $\mathfrak{su}(2)$

Suppose that $\alpha = \beta$ (the case $\alpha = -\beta$ is similar), so that

$$\iota_1 F = \alpha(e_{23} + e_{45}).$$

This means that $\iota_1 F$ belongs to the Cartan subalgebra of the self-dual SU(2) in SO(4). The condition $[\iota_1 F, F] = 0$ implies that only terms which have zero weights with respect to this self-dual $\mathfrak{su}(2)$ survive, whence

$$F = \alpha(e_{123} + e_{145}) + e_6 \wedge (\eta(e_{23} + e_{45}) + \gamma(e_{23} - e_{45}) + \delta(e_{34} - e_{25}) + \varepsilon(e_{24} + e_{35})).$$

However we are allowed to rotate the basis by the normaliser of this Cartan subalgebra, which is $U(1) \times SU(2)$, where the U(1) is the circle generated by $\iota_1 F$ and the SU(2) is anti-self-dual. Conjugating by the anti-self-dual SU(2) means that we can put $\delta = \varepsilon = 0$, say. The remaining equations $[\iota_X F, F] = 0$ are satisfied if and only if

$$\alpha^2 + \eta^2 = \gamma^2. \tag{8}$$

This means that

$$F = (\alpha e_1 + (\eta + \gamma)e_6) \wedge e_{23} + (\alpha e_1 + (\eta - \gamma)e_6) \wedge e_{45},$$

whence F can indeed be written as a sum of two simple 3-form which moreover are orthogonal since Eq. (8) implies that

$$(\alpha e_1 + (\eta + \gamma)e_6) \perp (\alpha e_1 + (\eta - \gamma)e_6),$$

as desired.

2.1.3. $\mathfrak{so}(2)$

Finally let us consider the case where

$$\iota_1 F = \alpha e_{23}.$$

The surviving terms in F after applying $[\iota_1 F, F] = 0$, are

$$F = \alpha e_{123} + \eta e_{234} + \gamma e_{235} + \delta e_{236} + \varepsilon e_{456}.$$

But we can rotate in the (456) plane to make $\gamma = \delta = 0$, whence

$$F = (\alpha e_1 + \eta e_4) \wedge e_{23} + \varepsilon e_4 \wedge e_{56}$$

can be written as a sum of two simple forms. Finally the remaining equations $[\iota_X F, F] = 0$ simply say that

$$\eta\varepsilon = 0, \tag{9}$$

whence the simple forms are orthogonal, since

$$(\alpha e_1 + \eta e_4) \perp \varepsilon e_4.$$

This verifies the conjecture for $d = 3$ and Euclidean signature.

2.2. Proof for $F \in \Lambda^3\mathbb{E}^{1,5}$

The Lorentzian case is almost identical to the Euclidean case, with a few signs in the equations distinguishing them. Let $F \in \Lambda^3\mathbb{E}^{1,5}$ be a 3-form in six-dimensional Minkowski space–time with pseudo-orthonormal basis $\{e_0, e_2, \ldots, e_6\}$ with e_0 time-like. Rotating if necessary in the five-dimensional Euclidean space spanned by $\{e_2, e_3, \ldots, e_6\}$, we can guarantee that

$$\iota_0 F = \alpha e_{23} + \beta e_{45},$$

as for the Euclidean case. As in that case, we must distinguish between three cases:

(1) $\mathfrak{so}(4)$: α and β generic,
(2) $\mathfrak{su}(2)$: $\alpha = \pm\beta \neq 0$, and
(3) $\mathfrak{so}(2)$: $\beta = 0, \alpha \neq 0$,

which we now briefly treat in turn.

In the first case, $[\iota_0 F, F] = 0$ means that the only terms in F which survive are

$$F = \alpha e_{023} + \beta e_{045} + \gamma e_{236} + \delta e_{456},$$

which is already a sum of two simple forms

$$F = (\alpha e_0 + \gamma e_6) \wedge e_{23} + (\beta e_0 + \delta e_6) \wedge e_{45}.$$

The remaining equations $[\iota_X F, F] = 0$ are satisfied if and only if

$$\alpha\beta = \gamma\delta, \tag{10}$$

which makes $\alpha e_0 + \gamma e_6$ and $\beta e_0 + \delta e_6$ orthogonal, verifying the conjecture in this case. We remark that this includes the null case as stated in [4] which corresponds to setting $\alpha = \beta = \gamma = \delta$.

In the second case, let $\iota_0 F = \alpha(e_{23} + e_{45})$, with the other possibility $\alpha = -\beta$ being similar. The equation $[\iota_0 F, F] = 0$ results in the following:

$$\begin{aligned} F = {} & \alpha(e_{023} + e_{045}) + e_6 \wedge (\eta(e_{23} + e_{45}) + \gamma(e_{23} - e_{45}) \\ & + \delta(e_{24} + e_{35}) + \varepsilon(e_{25} + e_{34})). \end{aligned}$$

We can rotate by the anti-self-dual $\mathrm{SU}(2) \subset \mathrm{SO}(4)$ in such a way that $\delta = \varepsilon = 0$, whence F take the desired form

$$F = (\alpha e_0 + (\eta + \gamma)e_6) \wedge e_{23} + (\alpha e_0 + (\eta - \gamma)e_6) \wedge e_{45}.$$

The remaining equations $[\iota_X F, F] = 0$ are satisfied if and only if

$$\alpha^2 + \gamma^2 = \eta^2, \tag{11}$$

which makes $\alpha e_0 + (\eta + \gamma)e_6$ and $\alpha e_0 + (\eta - \gamma)e_6$ orthogonal, verifying the conjecture in this case.

Finally let $\iota_0 F = \alpha e_{23}$. The equation $[\iota_0 F, F] = 0$ implies that

$$F = \alpha e_{023} + \eta e_{234} + \gamma e_{235} + \delta e_{236} + \varepsilon e_{456}.$$

Rotating in the (456) plane we can make $\gamma = \delta = 0$, whence F takes the desired form

$$F = (\alpha e_0 + \eta e_4) \wedge e_{23} + \varepsilon e_4 \wedge e_{56}.$$

The remaining equations $[\iota_X F, F] = 0$ are satisfied if and only if

$$\eta\varepsilon = 0, \qquad (12)$$

making $\alpha e_0 + \eta e_4$ and εe_4 orthogonal, and verifying the conjecture in this case, and hence in general for $d = 3$ and Lorentzian signature.

2.3. *Proof for* $F \in \Lambda^3\mathbb{E}^7$

Let $F \in \Lambda^3\mathbb{E}^7$ be a 3-form in a seven-dimensional Euclidean space with orthonormal basis $\{e_i\}_{i=1,\dots,7}$, relative to which the 2-form $\iota_7 F$ obtained by contracting e_7 into F takes the form

$$\iota_7 F = \alpha e_{12} + \beta e_{34} + \gamma e_{56},$$

where $e_{ij} = e_i \wedge e_j$ and similarly for $e_{ij\cdots k}$ in what follows.

We must distinguish several cases depending on whether α, β and γ are generic or not. In the general case, $\iota_7 F$ is a generic element of a Cartan subalgebra of $\mathfrak{so}(6)$ acting on the Euclidean space $\mathbb{E}^6$ spanned by $\{e_i\}_{i=1,\dots,6}$. The non-generic cases are in one-to-one correspondence with conjugacy classes of subalgebras of $\mathfrak{so}(6)$ of strictly lower rank. In summary we have the following cases to consider:

(1) $\mathfrak{so}(6)$: α, β and γ generic;
(2) $\mathfrak{su}(2) \times \mathfrak{u}(1)$: $\alpha = \pm\beta$ and γ generic;
(3) $\mathfrak{u}(1)$ diagonal: $\alpha = \beta = \gamma$;
(4) $\mathfrak{su}(3)$: $\alpha + \beta + \gamma = 0$;
(5) $\mathfrak{so}(4)$: α, β generic and $\gamma = 0$;
(6) $\mathfrak{su}(2)$: $\alpha = \pm\beta$ and $\gamma = 0$; and
(7) $\mathfrak{so}(2)$: $\gamma = \beta = 0$, $\alpha \neq 0$.

We now treat each case in turn.

2.3.1. $\mathfrak{so}(6)$

In the first case, α, β and γ are generic, whence the equation $[\iota_7 F, F] = 0$ says that only terms invariant under the maximal torus generated by $\iota_7 F$ survive, whence

$$F = \alpha e_{127} + \beta e_{347} + \gamma e_{567}.$$

The remaining equations $[\iota_i F, F] = 0$ are satisfied if and only if two of α, β and γ vanish, violating the hypothesis.

2.3.2. $\mathfrak{su}(2) \times \mathfrak{u}(1)$

We choose $\beta = \gamma$ and α generic. The case $\beta = -\gamma$ is similar. The equation $[\iota_7 F, F] = 0$ says that only terms invariant under the maximal torus generated by $\iota_7 F$ survive. Thus

$$F = \alpha e_{127} + \beta(e_{347} + e_{567}) + e_7 \wedge (\delta(e_{34} - e_{56}) + \varepsilon(e_{36} - e_{45}) + \eta(e_{25} + e_{46})).$$

Using an anti-self-dual rotation, we can set $\varepsilon = \eta = 0$. If $\delta \neq 0$, then $\beta + \delta \neq \beta - \delta$ and this leads to the case investigated in the previous section. If $\delta = 0$, invariance under $[\iota_1 F, F] = 0$ implies that either α or β vanishes, which violates the hypothesis.

2.3.3. $\mathfrak{u}(1)$ *diagonal*

Suppose that $\alpha = \beta = \gamma$. The equation $[\iota_7 F, F] = 0$ implies that

$$F = \alpha(e_{127} + e_{347} + e_{567}).$$

In addition invariance under $[\iota_1 F, F] = 0$ implies that $\alpha = 0$ which violates the hypothesis.

2.3.4. $\mathfrak{su}(3)$

Suppose that $\alpha + \beta + \gamma = 0$. The condition $[\iota_7 F, F] = 0$ implies that

$$F = (\alpha e_{127} + \beta e_{347} + \gamma e_{567}) + \delta \Omega_1 + \varepsilon \Omega_2,$$

where Ω_1 and the real and imaginary parts of the $\mathfrak{su}(3)$-invariant (3, 0)-form with respect to a complex structure $J = e_{12} + e_{34} + e_{56}$, that is,

$$\Omega_1 = e_{135} - e_{146} - e_{236} - e_{245}, \qquad \Omega_2 = e_{136} + e_{145} + e_{235} - e_{246}. \tag{13}$$

The presence of these forms can be seen from the decomposition of $\Lambda^3\mathbb{E}^6$ representation under $\mathfrak{su}(3)$. Under $\mathfrak{su}(3)$, the representation $\mathbb{E}^6$ transforms as the underlying real representation of $\mathbf{3} \oplus \bar{\mathbf{3}}$ (or $[\![\mathbf{3}]\!]$ in Salamon's notation [9]). Similarly the representation $\Lambda^3\mathbb{E}^6$ decomposes into

$$\Lambda^3\mathbb{E}^6 = [\![\mathbf{1}]\!] \oplus [\![\mathbf{6}]\!] \oplus [\![\mathbf{3}]\!].$$

The invariant forms are associated with the trivial representations in the decomposition. We still have the freedom to rotate by the normaliser in SO(6) of the maximal torus of SU(3). An obvious choice is the diagonal U(1) subgroup of U(3) which leaves invariant J. This U(1) rotates Ω_1 and Ω_2 and we can use it to set $\varepsilon = 0$. The new case is when $\delta \neq 0$. In such case invariance under the rest of the rotation $\iota_i F$ implies that $\alpha\beta + 2\delta^2 = 0$ and cyclic in α, β and γ. These relations contradict the hypothesis that $\alpha + \beta + \gamma = 0$ but otherwise generic.

2.3.5. $\mathfrak{so}(4)$

Suppose that α and β are generic and $\gamma = 0$. In that case, $[\iota_7 F, F] = 0$ implies that

$$F = \alpha e_{127} + \beta e_{347} + \delta_1 e_{125} + \delta_2 e_{126} + \varepsilon_1 e_{345} + \varepsilon_2 e_{346}.$$

Using a rotation in the (56) plane, we can set $\delta_2 = 0$. In addition δ_1 can also be set to zero with a rotation in the (57) plane and appropriate redefinition of the α, β and ε_1 components. Thus the 3-form can be written as

$$F = \alpha e_{127} + \beta e_{347} + \varepsilon_1 e_{345} + \varepsilon_2 e_{346}.$$

A rotation in the (56) plane leads to $\varepsilon_2 = 0$. The rest of the conditions $[\iota_i F, F] = 0$ imply that $\alpha\beta = 0$ which proves the conjecture.

2.3.6. $\mathfrak{su}(2)$

Suppose that $\alpha = \beta$ and $\gamma = 0$. The case $\alpha = -\beta$ can be treated similarly. The condition $[\iota_7 F, F] = 0$ implies that

$$\begin{aligned} F = {} & \alpha(e_{127} + e_{347}) + \delta(e_{125} + e_{345}) + \varepsilon(e_{126} + e_{346}) + \eta_1(e_{125} - e_{345}) \\ & + \eta_2(e_{145} - e_{235}) + \eta_3(e_{135} + e_{245}) + \theta_1(e_{126} - e_{346}) + \theta_2(e_{146} - e_{236}) \\ & + \theta_3(e_{136} + e_{246}). \end{aligned}$$

With an anti-self-dual rotation, we can set $\eta_2 = \eta_3 = 0$. There are two cases to consider. If $\eta_1 \neq 0$, the condition $[\iota_5 F, F] = 0$ implies that $\theta_2 = \theta_3 = 0$. In such case F can be rewritten as:

$$F = (\alpha e_7 + (\delta + \eta_1)e_5 + (\varepsilon + \theta_1)e_6) \wedge e_{12} + (\alpha e_7 + (\delta - \eta_1)e_5 + (\varepsilon - \theta_1)e_6) \wedge e_{34}.$$

The rest of the conditions imply that

$$\alpha^2 + \delta^2 - \eta_1^2 + \epsilon^2 - \theta_1^2 = 0$$

and so F is the sum of two orthogonal simple forms.

Now if $\eta_1 = 0$, an anti-self-dual rotation will give $\theta_2 = \theta_3 = 0$. This case is a special case of the previous one for which $\eta_1 = 0$. The conjecture is confirmed.

2.3.7. $\mathfrak{so}(2)$

Suppose that $\alpha \neq 0$ and $\beta = \gamma = 0$. The condition $[\iota_7 F, F] = 0$ implies that

$$F = \alpha e_{127} + \sigma_1 e_{123} + \sigma_2 e_{124} + \sigma_3 e_{125} + \sigma_4 e_{126} + \tau_1 e_{345} + \tau_2 e_{346} + \tau_3 e_{456}.$$

A rotation in the (3456) plane can lead to $\sigma_2 = \sigma_3 = \sigma_4 = 0$. If $\sigma_1 \neq 0$, then the condition $[\iota_1 F, F] = 0$ implies that $\tau_2 = \tau_1 = 0$ in which case

$$F = \alpha e_{127} + \sigma_1 e_{123} + \tau_3 e_{456}.$$

A further rotation in the (37) plane leads to the desired result.

Now if $\sigma_1 = 0$, a rotation in the (3456) plane can lead to $\tau_2 = \tau_3 = 0$ in which case

$$F = \alpha e_{127} + \tau_1 e_{345}.$$

This again gives the desired result.

2.4. Proof for $F \in \Lambda^3\mathbb{E}^d$ *and* $F \in \Lambda^3\mathbb{E}^{1,d-1}$, $d < 6$

We shall focus on the proof of the conjecture for $F \in \Lambda^3\mathbb{E}^d$. The proof of the statement in the Lorentzian case is similar. Let $F \in \Lambda^3\mathbb{E}^5$ be a 3-form in five-dimensional Euclidean space. There is an orthonormal basis $\{e_1, e_2, \ldots, e_5\}$ for which $\iota_1 F$ takes the form

$$\iota_1 F = \alpha e_{23} + \beta e_{45}.$$

As previous cases, there are several possibilities to consider depending on whether α and β are generic or not. Using the adopted group theoretic characterisation, we have the following cases:

(1) $\mathfrak{so}(4)$: α and β generic,
(2) $\mathfrak{su}(2)$: $\alpha = \pm\beta \neq 0$, and
(3) $\mathfrak{so}(2)$: $\beta = 0$, $\alpha \neq 0$.

We now treat each case in turn.

2.4.1. $\mathfrak{so}(4)$

In the first case, α and β are generic, whence the equation $[\iota_1 F, F] = 0$ says that only terms invariant under the maximal torus generated by $\iota_1 F$ survive, whence

$$F = \alpha e_{123} + \beta e_{145}$$

The remaining equations $[\iota_i F, F] = 0$ are satisfied if and only if

$$\alpha\beta = 0, \tag{14}$$

which is a contradiction. Thus $\iota_1 F$ cannot be generic.

2.4.2. $\mathfrak{su}(2)$

Suppose that $\alpha = \beta$ (the case $\alpha = -\beta$ is similar), so that

$$\iota_1 F = \alpha(e_{23} + e_{45}).$$

This means that $\iota_1 F$ belongs to the Cartan subalgebra of the self-dual SU(2) in SO(4). The condition $[\iota_1 F, F] = 0$ implies that only terms which have zero weights with respect to this self-dual $\mathfrak{su}(2)$ survive, and so

$$F = \alpha(e_{123} + e_{145}).$$

The remaining equations $[\iota_X F, F] = 0$ are satisfied if and only if

$$\alpha^2 = 0, \tag{15}$$

which is a contradiction. Thus $\iota_1 F$ cannot be self-dual.

2.4.3. $\mathfrak{so}(2)$

Finally let us consider the case where

$$\iota_1 F = \alpha e_{23}.$$

The surviving terms in F after applying $[\iota_1 F, F] = 0$, are

$$F = \alpha e_{123} + \eta e_{234} + \gamma e_{235}.$$

But we can rotate in the (45) plane to make $\gamma = 0$, whence

$$F = (\alpha e_1 + \eta e_4) \wedge e_{23}$$

is a simple form. This verifies the conjecture for $d = 5$ and Euclidean signature.

2.4.4. Proof for $F \in \Lambda^3\mathbb{E}^d$ and $F \in \Lambda^3\mathbb{E}^{1,d-1}$, $d = 3, 4$

The proof for $d = 3$ is obvious. It remains to show the conjecture for $d = 4$. In Euclidean signature, we have

$$\iota_1 F = \alpha e_{23}.$$

The surviving terms in F after applying $[\iota_1 F, F] = 0$, are

$$F = \alpha e_{123} + \eta e_{234}.$$

which can be rewritten as

$$F = (\alpha e_1 + \eta e_4) \wedge e_{23}$$

and so it is a simple form. This verifies the conjecture for $d = 4$ and Euclidean signature. The proof for Lorentzian spaces is similar.

2.5. Metric Lie algebras and the case $p = 3$

We can give an alternate proof for the case $p = 3$ exploiting the relationship with metric Lie algebras; that is, Lie algebras admitting an invariant non-degenerate scalar product.

It is well known that reductive Lie algebras—that is, direct products of semisimple and Abelian Lie algebras—admit invariant scalar products: Cartan's criterion allows us to use the Killing form on the semisimple factor and any scalar product on an Abelian Lie algebra is automatically invariant.

Another well-known example of Lie algebras admitting an invariant scalar product are the classical doubles. Let $\mathfrak{h}$ be *any* Lie algebra and let $\mathfrak{h}^*$ denote the dual space on which $\mathfrak{h}$ acts via the coadjoint representation. The definition of the coadjoint representation is such that the dual pairing $\mathfrak{h} \otimes \mathfrak{h}^* \to \mathbb{R}$ is an invariant scalar product on the semidirect product $\mathfrak{h} \ltimes \mathfrak{h}^*$ with $\mathfrak{h}^*$ an Abelian ideal. The Lie algebra $\mathfrak{h} \ltimes \mathfrak{h}^*$ is called the classical double of $\mathfrak{h}$ and the invariant metric has split signature (r, r) where $\dim \mathfrak{h} = r$.

It turns out that all Lie algebras admitting an invariant scalar product can be obtained by a mixture of these constructions. Let $\mathfrak{g}$ be a Lie algebra with an invariant scalar product $\langle -, - \rangle_{\mathfrak{g}}$, and let $\mathfrak{h}$ act on $\mathfrak{g}$ preserving both the Lie bracket and the scalar product; in other words, $\mathfrak{h}$ acts on $\mathfrak{g}$ via skew-symmetric derivations. First of all, since $\mathfrak{h}$ acts on $\mathfrak{g}$ preserving the scalar product, we have a linear map

$$\mathfrak{h} \to \mathfrak{so}(\mathfrak{g}) \cong \Lambda^2\mathfrak{g}$$

with dual map

$$c : \Lambda^2\mathfrak{g}^* \cong \Lambda^2\mathfrak{g} \to \mathfrak{h}^*,$$

where we have used the invariant scalar product to identity $\mathfrak{g}$ and $\mathfrak{g}^*$ equivariantly. Since $\mathfrak{h}$ preserves the Lie bracket in $\mathfrak{g}$, this map is a cocycle, whence it defines a class $[c] \in H^2(\mathfrak{g}; \mathfrak{h}^*)$ in the second Lie algebra cohomology of $\mathfrak{g}$ with coefficients in the trivial module $\mathfrak{h}^*$. Let $\mathfrak{g} \times_c \mathfrak{h}^*$ denote the corresponding central extension. The Lie bracket of the $\mathfrak{g} \times_c \mathfrak{h}^*$ is such that $\mathfrak{h}^*$ is central and if $X, Y \in \mathfrak{g}$, then

$$[X, Y] = [X, Y]_{\mathfrak{g}} + c(X, Y),$$

where $[-,-]_{\mathfrak{g}}$ is the Lie bracket of $\mathfrak{g}$. Now $\mathfrak{h}$ acts naturally on this central extension: the action on $\mathfrak{h}^*$ given by the coadjoint representation. This then allows us to define the *double extension* of $\mathfrak{g}$ by $\mathfrak{h}$,

$$\mathfrak{d}(\mathfrak{g},\mathfrak{h}) = \mathfrak{h} \ltimes (\mathfrak{g} \times_c \mathfrak{h}^*)$$

as a semidirect product. Details of this construction can be found in [10,11]. The remarkable fact is that $\mathfrak{d}(\mathfrak{g},\mathfrak{h})$ admits an invariant inner product:

$$\begin{array}{c} \\ \mathfrak{g} \\ \mathfrak{h} \\ \mathfrak{h}^* \end{array} \begin{array}{c} \begin{array}{ccc} \mathfrak{g} & \mathfrak{h} & \mathfrak{h}^* \end{array} \\ \begin{pmatrix} \langle -,- \rangle_{\mathfrak{g}} & 0 & 0 \\ 0 & B & \mathrm{id} \\ 0 & \mathrm{id} & 0 \end{pmatrix} \end{array} \tag{16}$$

where B is *any* invariant symmetric bilinear form on $\mathfrak{h}$ and id stands for the dual pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$.

We say that a Lie algebra with an invariant scalar product is indecomposable if it cannot be written as the direct product of two orthogonal ideals. A theorem of Medina and Revoy [10] (see also [12] for a refinement) says that an indecomposable (finite-dimensional) Lie algebra with an invariant scalar product is one of the following:

(1) one-dimensional,
(2) simple, or
(3) a double extension $\mathfrak{d}(\mathfrak{g},\mathfrak{h})$, where $\mathfrak{h}$ is either simple or one-dimensional and $\mathfrak{g}$ is a Lie algebra with an invariant scalar product. (Notice that we can take $\mathfrak{g}$ to be the trivial zero-dimensional Lie algebra. In this way we recover the classical double.)

Any (finite-dimensional) Lie algebra with an invariant scalar product is then a direct sum of indecomposables.

Notice that if the scalar product on $\mathfrak{g}$ has signature (p,q) and if $\dim\mathfrak{h} = r$, then the scalar product on $\mathfrak{d}(\mathfrak{g},\mathfrak{h})$ has signature $(p+r, q+r)$. Therefore Euclidean Lie algebras are necessarily reductive, and if indecomposable they are either one-dimensional or simple. Up to dimension 7 we have the following Euclidean Lie algebras:

- $\mathbb{R}^d$ with $d \le 7$,
- $\mathfrak{su}(2) \oplus \mathbb{R}^k$ with $k \le 4$, and
- $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^k$ with $k = 0, 1$.

The conjecture clearly holds for all of them.

The Lorentzian case is more involved. Indecomposable Lorentzian Lie algebras are either reductive or double extensions $\mathfrak{d}(\mathfrak{g},\mathfrak{h})$, where $\mathfrak{g}$ has a positive-definite invariant scalar product and $\mathfrak{h}$ is one-dimensional. In the reductive case, indecomposability means that it has to be simple, whereas in the latter case, since the scalar product on $\mathfrak{g}$ is positive-definite, $\mathfrak{g}$ must be reductive. A result of Figueroa-O'Farrill and Stanciu [11] (see also [12]) then says that any semisimple factor in $\mathfrak{g}$ splits off resulting in a decomposable Lie algebra. Thus if the double extension is to be indecomposable, then $\mathfrak{g}$ must be Abelian. In summary, an indecomposable Lorentzian Lie algebra is either simple or a double extension of an Abelian Lie algebra by a one-dimensional Lie algebra and hence solvable (see, e.g., [10]).

These considerations make possible the following enumeration of Lorentzian Lie algebras up to dimension 7:

(1) $\mathbb{E}^{1,d-1}$ with $d \le 7$,
(2) $\mathbb{E}^{1,k} \oplus \mathfrak{so}(3)$ with $k \le 3$,
(3) $\mathbb{E}^{k} \oplus \mathfrak{so}(1,2)$ with $k \le 4$,
(4) $\mathfrak{so}(1,2) \oplus \mathfrak{so}(3) \oplus \mathbb{E}^{k}$ with $k = 0, 1$, or
(5) $\mathfrak{d}(\mathbb{E}^4, \mathbb{R}) \oplus \mathbb{E}^k$ with $k = 0, 1$,

where the last case actually corresponds to a family of Lie algebras, depending on the action of $\mathbb{R}$ on $\mathbb{E}^4$. The conjecture holds manifestly for all cases except possibly the last, which we must investigate in more detail.

Let e_i, $i = 1, 2, 3, 4$, be an orthonormal basis for $\mathbb{E}^4$, and let $e_- \in \mathbb{R}$ and $e_+ \in \mathbb{R}^*$, so that together they span $\mathfrak{d}(\mathbb{E}^4, \mathbb{R})$. The action of $\mathbb{R}$ on $\mathbb{R}^4$ defines a map $\rho : \mathbb{R} \to \Lambda^2\mathbb{R}^4$, which can be brought to the form $\rho(e_-) = \alpha e_1 \wedge e_2 + \beta e_3 \wedge e_4$ via an orthogonal change of basis in $\mathbb{E}^4$ which moreover preserves the orientation. The Lie brackets of $\mathfrak{d}(\mathbb{E}^4, \mathbb{R})$ are given by

$$
\begin{aligned}
&[e_-, e_1] = \alpha e_2, \qquad &&[e_-, e_2] = -\alpha e_1,\\
&[e_1, e_2] = \alpha e_+, \qquad &&[e_-, e_3] = \beta e_4,\\
&[e_-, e_4] = -\beta e_3, \quad &&[e_3, e_4] = \beta e_+,
\end{aligned}
$$

and the scalar product is given (up to scale) by

$$
\langle e_-, e_-\rangle = b, \quad \langle e_+, e_-\rangle = 1, \quad \langle e_i, e_j\rangle = \delta_{ij}.
$$

The first thing we notice is that we can set $b = 0$ without loss of generality by the automorphism fixing all e_i, e_+ and mapping $e_- \mapsto e_- - (1/2)be_+$. We will assume that this has been done and that $\langle e_-, e_-\rangle = 0$. A straightforward calculation shows that the 3-form F takes the form

$$
F = \alpha e_- \wedge e_1 \wedge e_2 + \beta e_- \wedge e_3 \wedge e_4,
$$

whence the conjecture holds.

2.6. *Proof for $F \in \Lambda^4\mathbb{E}^8$*

In the absence (to our knowledge) of a structure theorem for metric n-Lie algebras, we will present the verification of the conjecture in the remaining cases using the "brute-force" approach explained earlier.

Choose an orthonormal basis $\{e_1, e_2, \ldots, e_8\}$ for which $\iota_{12}F = \alpha e_{34} + \beta e_{56} + \gamma e_{78}$, where ι_{12} means the contraction of F by e_{12}.

Suppose that α, β and γ are generic. In this case, the equation $[\iota_{12}F, F] = 0$ says that the only terms in F which survive are those which are invariant under the maximal torus of SO(6), the group of rotations in the six-dimensional space spanned by $\{e_3, e_4, \ldots, e_8\}$; that is,

$$
F = \alpha e_{1234} + \beta e_{1256} + \gamma e_{1278} + \delta e_{3456} + \varepsilon e_{3478} + \eta e_{5678}.
$$

Now, $\iota_{13}F = -\alpha e_{24}$, whence the equation $[\iota_{13}F, F] = 0$ implies that $\beta = \gamma = \delta = \varepsilon = 0$, violating the condition that $\iota_{12}F$ be generic.

In fact, this argument clearly works for $d \geq 4$ so that for $d \geq 4$ we have to deal with non-generic rotations. Non-generic rotations correspond to (conjugacy classes of) subalgebras of $\mathfrak{so}(6)$ with rank strictly less than that of $\mathfrak{so}(6)$:

(1) $\mathfrak{su}(3)$: $\alpha + \beta + \gamma = 0$ but all α, β, and γ non-zero;
(2) $\mathfrak{su}(2) \times \mathfrak{u}(1)$: $\alpha = \beta \neq \gamma$, but again all non-zero;
(3) $\mathfrak{u}(1)$ diagonal: $\alpha = \beta = \gamma \neq 0$;
(4) $\mathfrak{so}(4)$: $\gamma = 0$ and $\alpha \neq \beta$ non-zero;
(5) $\mathfrak{su}(2)$: $\gamma = 0$ and $\alpha = \beta \neq 0$; and
(6) $\mathfrak{so}(2)$: $\beta = \gamma = 0$ and $\alpha \neq 0$.

We now go down this list case by case.

2.6.1. $\mathfrak{su}(3)$

When $\iota_{12}F$ is a generic element of the Cartan subalgebra of an $\mathfrak{su}(3)$ subalgebra of $\mathfrak{so}(6)$ the only terms in F which satisfy the equation $[\iota_{12}F, F] = 0$ are those which have zero weights relative to this Cartan subalgebra. Let $\mathbb{E}^6 = \langle e_1, e_2\rangle^\perp$. Then F can be written as

$$F = e_{12} \wedge \iota_{12}F + G,$$

where G is in the kernel of ι_{12}, namely

$$G = e_1 \wedge G_1 + e_2 \wedge G_2 + G_3,$$

where $G_1, G_2 \in \Lambda^3\mathbb{E}^6$ and $G_3 \in \Lambda^4\mathbb{E}^6$. We have investigated the decomposition of $\Lambda^3\mathbb{E}^6$ under $\mathfrak{su}(3)$ in the previous section. The representation $\Lambda^4\mathbb{E}^6$ decomposes into

$$\Lambda^4\mathbb{E}^6 = \mathbf{1} \oplus \mathbf{8} \oplus [\![\mathbf{3}]\!],$$

whence it is clear where the zero weights are: they are one in the trivial representation **1** and two in the adjoint **8**. This means that in this case together with the zero weights of the $\Lambda^3\mathbb{E}^6$ representations a total of seven terms in G:

$$G_1 = \lambda_1\Omega_1 + \lambda_2\Omega_2, \quad G_2 = \lambda_3\Omega_1 + \lambda_4\Omega_2, \quad G_3 = \mu_1 e_{3456} + \mu_2 e_{3478} + \mu_3 e_{5678},$$

where

$$\Omega_1 = e_{357} - e_{368} - e_{458} - e_{467}, \qquad \Omega_2 = e_{358} + e_{367} + e_{457} - e_{468} \tag{17}$$

are the real and imaginary parts, respectively, of the holomorphic 3-form in $\mathbb{E}^6$ thought of as $\mathbb{C}^3$ with the $\mathfrak{su}(3)$-invariant complex structure $J = e_{34} + e_{56} + e_{78}$. We still have to freedom to rotate by the normaliser in SO(6) of the maximal torus in SU(3) that $\iota_{12}F$ determines. An obvious choice is the U(1) generated by the complex structure. This is not in SU(3) but in U(3) and has the virtue of acting on $\Omega = \Omega_1 + \mathrm{i}\Omega_2$ by multiplication by a complex phase. This means that we can always choose Ω to be real, thus setting $\lambda_4 = 0$, say. Analysing the remaining equations $[\iota_{ij}F, F] = 0$ we see that α and β are constrained to $\alpha = \pm\beta$, violating the hypothesis that they are generic.

2.6.2. $\mathfrak{su}(2) \times \mathfrak{u}(1)$

Let us consider $\alpha = \beta$, the other case being similar, in fact related by conjugation in O(4), which is an outer automorphism. The equation $[\iota_{12}F, F] = 0$ says that the only terms in F which survive are those corresponding to zero weights of the $\mathfrak{su}(2) \times \mathfrak{u}(1)$ subalgebra of $\mathfrak{so}(6)$. It is easy to see that $\Lambda^3\mathbb{E}^6$ has non-zero weights, whereas the zero weights in $\Lambda^4\mathbb{E}^6$ are the Hodge duals of the following 2-forms:

$$e_{34}, \quad e_{56}, \quad e_{78}, \quad e_{35} + e_{46}, \quad e_{36} - e_{45}.$$

Conjugating by the anti-self-dual SU(2) we can set to zero the coefficients of the last two forms, leaving

$$F = \alpha(e_{1234} + e_{1256}) + \gamma e_{1278} + \mu_1 e_{3456} + \mu_2 e_{3478} + \mu_3 e_{5678}$$

as the most general solution of $[\iota_{12}F, F] = 0$. Now the equation $[\iota_{13}F, F] = 0$, for example, implies that α must vanish, violating the hypothesis. This case is therefore discarded.

2.6.3. $\mathfrak{u}(1)$ *diagonal*

In this case, $\iota_{12}F = \alpha(e_{34} + e_{56} + e_{78})$ belongs to the diagonal $\mathfrak{u}(1)$ which is the centre of $\mathfrak{u}(3) \subset \mathfrak{so}(6)$, where $\mathfrak{so}(6)$ acts on the $\mathbb{E}^6$ spanned by $\{e_i\}_{3\leq i\leq 8}$. There are no zero weights in $\Lambda^3\mathbb{E}^6$, but there are nine in $\Lambda^4\mathbb{E}^6$: the Hodge duals of $\mathfrak{u}(3) \subset \mathfrak{so}(6) \cong \Lambda^2\mathbb{E}^6$. However we are allowed to conjugate by the normaliser of $\mathfrak{u}(1)$ in $\mathfrak{so}(6)$ which is $\mathfrak{u}(3)$. This allows us to conjugate the invariant 2-forms to lie in the Cartan subalgebra of $\mathfrak{u}(3)$. In summary, the solution to $[\iota_{12}F, F] = 0$ can be written in the form

$$F = \alpha(e_{1234} + e_{1256} + e_{1278}) + \mu_1 e_{3456} + \mu_2 e_{3478} + \mu_3 e_{5678}.$$

Now we consider, for example, the equation $[\iota_{13}F, F] = 0$ and we see that α must vanish, violating the hypothesis. Thus this case is also discarded.

Notice that all the cases where the 2-form $\iota_{12}F$ has maximal rank have been discarded, often after a detailed analysis of the equations. This should have a simpler explanation.

2.6.4. $\mathfrak{so}(4)$

In this case $\iota_{12}F = \alpha e_{34} + \beta e_{56}$, where α and β are generic. This means that the most general solution of $[\iota_{12}F, F] = 0$ is given by

$$F = \alpha e_{1234} + \beta e_{1256} + G,$$

where G is of the form $e_1 \wedge G_1 + e_2 \wedge G_2 + G_3$, where $G_1, G_2 \in \Lambda^3\mathbb{E}^6$ and $G_3 = \Lambda^4\mathbb{E}^6$, where $\mathbb{E}^6$ is spanned by $\{e_i\}_{3\leq i\leq 8}$, and where the G_i have zero weight with respect to this $\mathfrak{so}(4)$ algebra. A little group theory shows that G_1 and G_2 are linear combinations of the four monomials $e_{347}, e_{348}, e_{567}, e_{568}$; whereas G_3 is a linear combination of the three monomials $e_{3456}, e_{3478}, e_{5678}$. We still have the freedom to conjugate by the normaliser in SO(6) of the maximal torus generated by $\iota_{12}F$, which includes the SO(2) of rotations in the (78) plane. Doing this we can set any one of the monomials in $e_1 \wedge G_1$, say e_{1347}, to zero. In summary, the most general solution of $[\iota_{12}F, F] = 0$ can be put in the following form:

$$F = \alpha e_{1234} + \beta e_{1256} + \mu_1 e_{3456} + \mu_2 e_{3478} + \mu_3 e_{5678} + \lambda_1 e_{1348} + \lambda_2 e_{1567} + \lambda_3 e_{1568} + \lambda_4 e_{2347} + \lambda_5 e_{2348} + \lambda_6 e_{2567} + \lambda_7 e_{2568}.$$

Analysing the remaining equations $[\iota_{ij}F, F] = 0$ we notice that genericity of α and β are violated unless $\mu_1 = 0$ and $\mu_3\mu_2 = \alpha\beta$. Given this we find that the most general solution is

$$F = \alpha e_{1234} + \beta e_{1256} + \mu_3 e_{5678} + \mu_2 e_{3478} + \nu_1(\alpha e_{1348} + \mu_3 e_{2567}) + \nu_2(\beta e_{1567} - \mu_2 e_{2348}) + \nu_3(\beta e_{1568} + \mu_2 e_{2347})$$

subject to

$$\nu_1\nu_3 = -1 \quad \text{and} \quad \mu_3\mu_2 = \alpha\beta. \tag{18}$$

These identities are precisely the ones that allow us to rewrite F as a sum of two simple forms

$$F_1 = (\alpha e_1 - \mu_2(\nu_3 e_7 - \nu_2 e_8)) \wedge (e_2 + \nu_1 e_8) \wedge e_3 \wedge e_4,$$
$$F_2 = (\beta e_1 - \mu_3\nu_1 e_7) \wedge (e_2 + \nu_2 e_7 + \nu_3 e_8) \wedge e_5 \wedge e_6,$$

which moreover are orthogonal.

2.6.5. $\mathfrak{su}(2)$

In this case $\iota_{12}F = \alpha(e_{34} + e_{56})$, where without loss of generality we can set $\alpha = 1$. This means that the most general solution of $[\iota_{12}F, F] = 0$ is given by

$$F = e_{1234} + e_{1256} + e_1 \wedge G_1 + e_2 \wedge G_2 + G_3,$$

where $G_1, G_2 \in \Lambda^3\mathbb{E}^6$ and $G_3 = \Lambda^4\mathbb{E}^6$, where $\mathbb{E}^6$ is spanned by $\{e_i\}_{3\le i\le 8}$, and where the G_i have zero weight with respect to this $\mathfrak{su}(2)$ algebra. A little group theory shows that G_1 and G_2 are linear combinations of the following eight 3-forms:

$$e_{34i} + e_{56i}, \quad e_{34i} - e_{56i}, \quad e_{35i} + e_{46i}, \quad e_{36i} - e_{45i},$$

where i can be either 7 or 8; whereas G_3 is the Hodge dual (in $\mathbb{E}^6$) of a linear combination of

$$e_{34} + e_{56}, \quad e_{34} - e_{56}, \quad e_{35} + e_{46}, \quad e_{36} - e_{45}.$$

Using the freedom to conjugate by the normaliser of $\mathfrak{su}(2)$ in $\mathfrak{so}(6)$ we can choose basis such that G_3 takes the form

$$G_3 = \mu_1 e_{3456} + \mu_2 e_{3478} + \mu_3 e_{5678}.$$

This means that F takes the following form:

$$\begin{aligned} F = {} & e_{1234} + e_{1256} + \mu_1 e_{3456} + \mu_2 e_{3478} + \mu_3 e_{5678} + \lambda_1 e_{1347} + \lambda_2 e_{1348} + \lambda_3 e_{1567} \\ & + \lambda_4 e_{1568} + \lambda_5 e_{2347} + \lambda_6 e_{2348} + \lambda_7 e_{2567} + \lambda_8 e_{2568} + \sigma_1(e_{1357} + e_{1467}) \\ & + \sigma_2(e_{1367} - e_{1457}) + \sigma_3(e_{1358} + e_{1468}) + \sigma_4(e_{1368} - e_{1458}) \\ & + \sigma_5(e_{2357} + e_{2467}) + \sigma_6(e_{2367} - e_{2457}) + \sigma_7(e_{2358} + e_{2468}) \\ & + \sigma_8(e_{2368} - e_{2458}). \end{aligned}$$

This still leaves the possibility of rotating, for example, in the (78) plane and an anti-self-dual rotation in the (3456) plane. Rotating in the (78) plane allows us to set $\lambda_8 = 0$, whereas an anti-self-dual rotation allows us to set $\sigma_8 = 0$. Imposing, for example, the equation $[\iota_{25}F, F] = 0$ tells us that $\lambda_1 = 0$, whereas the rest of the equations also say that $\sigma_2 = 0$. It follows after a little work that if $\mu_1 \neq 0$ we arrive at a contradiction, so that we take $\mu_1 = 0$.

We now have to distinguish between two cases, depending on whether or not μ_2 equals μ_3. If $\mu_2 \neq \mu_3$, then all $\sigma_i = 0$, and moreover F takes the form

$$\begin{aligned} F &= e_{1234} + e_{1256} + \mu_2 e_{3478} + \mu_3 e_{5678} + \lambda_2(e_{1348} + \mu_3 e_{2567}) \\ &\quad + \lambda_3(e_{1567} - \mu_2 e_{2348}) + \lambda_4(e_{1568} + \mu_2 e_{2347}), \end{aligned}$$

subject to the equations

$$\lambda_2\lambda_4 = -1 \quad \text{and} \quad \mu_2\mu_3 = 1. \tag{19}$$

These equations are precisely what is needed to write F as a sum of two orthogonal simple forms $F = F_1 + F_2$, where

$$\begin{aligned} F_1 &= (e_1 - \mu_2(\lambda_4 e_7 - \lambda_3 e_8)) \wedge (e_2 + \lambda_2 e_8) \wedge e_3 \wedge e_4 F_2 \\ &= (e_1 - \mu_3\lambda_2 e_7) \wedge (e_2 + \lambda_3 e_7 + \lambda_4 e_8) \wedge e_5 \wedge e_6. \end{aligned}$$

Finally, we consider the case $\mu_2 = \mu_3$, which is inconsistent unless $\mu_2^2 = 1$. Then the most general solution takes the form

$$\begin{aligned} F &= e_{1234} + e_{1256} + \mu_2(e_{3478} + e_{5678}) + \lambda_2(e_{1348} + \mu_2 e_{2567}) \\ &\quad + \lambda_3(e_{1567} - \mu_2 e_{2348}) + \lambda_4(e_{1568} + \mu_2 e_{2347}) \\ &\quad + \sigma_1(e_{1357} + e_{1467} + \mu_2 e_{2358} + \mu_2 e_{2468}) \\ &\quad + \sigma_3(e_{1358} + e_{1468} - \mu_2 e_{2357} - \mu_2 e_{2467}) \\ &\quad + \sigma_4(e_{1368} - e_{1458} - \mu_2 e_{2367} + \mu_2 e_{2457}), \end{aligned}$$

subject to the following equations:

$$\lambda_3\sigma_4 = 0 = \sigma_1\sigma_4, \qquad (\lambda_2 - \lambda_4)\sigma_1 + \lambda_3\sigma_3 = 0, \qquad \sigma_1^2 + \sigma_3^2 + \sigma_4^2 = 1 + \lambda_2\lambda_4. \tag{20}$$

Let us rewrite F in terms of (anti)self-dual 2-forms in the (1278) and (3456) planes:

$$\begin{aligned} F &= [(e_{12} + \mu_2 e_{78}) + \tfrac{1}{2}\lambda_3(e_{17} - \mu_2 e_{28}) + \tfrac{1}{2}(\lambda_2 + \lambda_4)(e_{18} + \mu_2 e_{27})] \\ &\quad \wedge(e_{34} + e_{56}) + (e_{17} + \mu_2 e_{28}) \wedge [\sigma_1(e_{35} + e_{46}) - \tfrac{1}{2}\lambda_3(e_{34} - e_{56})] \\ &\quad + (e_{18} - \mu_2 e_{27}) \wedge [\sigma_3(e_{35} + e_{46}) + \sigma_4(e_{36} - e_{45}) + \tfrac{1}{2}(\lambda_2 - \lambda_4)(e_{34} - 3_{56})]. \end{aligned}$$

Notice that the first two equations in (20) simply say that the two anti-self-dual 2-forms

$$\begin{aligned} &\sigma_1(e_{35} + e_{46}) - \tfrac{1}{2}\lambda_3(e_{34} - e_{56}), \\ &\sigma_3(e_{35} + e_{46}) + \sigma_4(e_{36} - e_{45}) + \tfrac{1}{2}(\lambda_2 - \lambda_4)(e_{34} - 3_{56}) \end{aligned}$$

are collinear. Therefore performing an anti-self-dual rotation in the (36)–(45) direction, we can eliminate the $e_{35}+e_{46}$ and $e_{36}-e_{45}$ components, effectively setting $\sigma_1 = \sigma_3 = \sigma_4 = 0$. This reduces the problem to the previous case, except that now $\mu_2 = \mu_3$.

2.6.6. $\mathfrak{so}(2)$

Finally, we consider the case where $\iota_{12}F = \alpha e_{34}$. The most general F has the form

$$F = \alpha e_{1234} + e_1 \wedge G_1 + e_2 \wedge G_2 + G_3,$$

where $G_1, G_2 \in \Lambda^3\mathbb{E}^6$ and $G_3 \in \Lambda^4\mathbb{E}^6$, where $\mathbb{E}^6$ is spanned by $\{e_i\}_{3\leq i\leq 8}$. Such an F will obey $[\iota_{12}F, F] = 0$ if and only if the G_i have zero weights under the $\mathfrak{so}(2)$ generated by $\iota_{12}F$. This means that each of G_1, G_2 is a linear combination of the eight monomials

$$e_{345}, \quad e_{346}, \quad e_{347}, \quad e_{348}, \quad e_{567}, \quad e_{568}, \quad e_{578}, \quad e_{678}.$$

Using the freedom to conjugate by the SO(4) which acts in the (5678) plane, we can write the most general G_3 as a linear combination of the monomials $e_{5678}, e_{3478}, e_{3456}$. This still leaves the possibility of rotating in the (56)- and (78) planes separately. Doing so we can set to zero the coefficients of say, e_{2568} and e_{2678}, leaving a total of 17 free parameters

$$\begin{aligned} F = {} & e_{1234} + \mu_1 e_{3456} + \mu_2 e_{3478} + \mu_3 e_{5678} + \lambda_1 e_{1347} + \lambda_2 e_{1348} + \lambda_3 e_{1567} + \lambda_4 e_{1568} \\ & + \lambda_5 e_{2347} + \lambda_6 e_{2348} + \lambda_7 e_{2567} + \sigma_1 e_{1345} + \sigma_2 e_{1346} + \sigma_3 e_{1578} + \sigma_4 e_{1678} \\ & + \sigma_5 e_{2345} + \sigma_6 e_{2346} + \sigma_7 e_{2578}, \end{aligned}$$

and where we have set $\alpha = 1$ without loss of generality. We now impose the rest of the equations $[\iota_{ij}F, F] = 0$. We first observe that if $\mu_1 \neq 0$, then $\mu_2 = \mu_3 = \lambda_i = \sigma_3 = \sigma_4 = \sigma_7 = 0$, leaving

$$F = e_{1234} + \mu_1 e_{3456} + \sigma_1 e_{1345} + \sigma_2 e_{1346} + \sigma_5 e_{2345} + \sigma_6 e_{2346},$$

subject to

$$\sigma_1\sigma_6 - \sigma_2\sigma_5 = \mu_1, \tag{21}$$

which guarantees that F is actually a simple form

$$F = (e_1 - \sigma_5 e_5 - \sigma_6 e_6) \wedge (e_2 + \sigma_1 e_5 + \sigma_2 e_6) \wedge e_3 \wedge e_4,$$

which is a degenerate case of the conclusion of the conjecture.

Let us then suppose that $\mu_1 = 0$. We next observe that if $\mu_2 \neq 0$ then $\mu_3 = \sigma_i = \lambda_3 = \lambda_4 = \lambda_7 = 0$. This is again, up to a relabelling of the coordinates, the same degenerate case as before and the conclusion still holds.

Finally let us suppose that both μ_1 and μ_2 vanish. We must distinguish between two cases, depending on whether μ_3 also vanishes or not. If $\mu_3 = 0$ then we have that F is given by

$$\begin{aligned} F = {} & e_{1234} + \lambda_1 e_{1347} + \lambda_2 e_{1348} + \lambda_5 e_{2347} + \lambda_6 e_{2348} + \sigma_1 e_{1345} + \sigma_2 e_{1346} \\ & + \sigma_5 e_{2345} + \sigma_6 e_{2346}, \end{aligned}$$

subject to the equations

$$\begin{aligned} &\lambda_2\lambda_5 = \lambda_1\lambda_6, \quad \sigma_2\sigma_5 = \sigma_1\sigma_6,\\ &\lambda_1\sigma_5 = \lambda_5\sigma_1, \quad \lambda_6\sigma_2 = \lambda_2\sigma_6,\\ &\lambda_1\sigma_6 = \lambda_5\sigma_2, \quad \lambda_6\sigma_1 = \lambda_2\sigma_5, \end{aligned} \tag{22}$$

which are precisely the equations which allow us to rewrite F as a simple form $F = \theta_1 \wedge \theta_2 \wedge e_3 \wedge e_4$, where

$$\theta_1 = e_1 - \sigma_5 e_5 - \sigma_6 e_6 - \lambda_5 e_7 - \lambda_6 e_8, \qquad \theta_2 = e_2 + \sigma_1 e_5 + \sigma_2 e_6 + \lambda_1 e_7 + \lambda_2 e_8.$$

Finally suppose that $\mu_3 \neq 0$. In this case F is given by

$$\begin{aligned} F = {}& e_{1234} + \mu_3 e_{5678} + \lambda_2(e_{1348} + \mu_3 e_{2567}) + \lambda_5(e_{2347} + \mu_3 e_{1568})\\ &+ \lambda_6(e_{2348} - \mu_3 e_{1567}) + \sigma_2(e_{1346} + \mu_3 e_{2578}) + \sigma_5(e_{2345} + \mu_3 e_{1678})\\ &+ \sigma_6(e_{2346} - \mu_3 e_{1578}), \end{aligned}$$

subject to the equations

$$\lambda_2\lambda_5 = \lambda_2\sigma_5 = \sigma_2\lambda_5 = \sigma_2\sigma_5 = 0 \quad \text{and} \quad \lambda_6\sigma_2 = \lambda_2\sigma_6. \tag{23}$$

We must distinguish between three cases:

(1) $\lambda_2 \neq 0$,
(2) $\lambda_2 = 0$ and $\sigma_2 \neq 0$, and
(3) $\lambda_2 = \sigma_2 = 0$.

We now do each in turn.

If $\lambda_2 \neq 0$, F is given by

$$\begin{aligned} F = {}& e_{1234} + \mu_3 e_{5678} + \lambda_2(e_{1348} + \mu_3 e_{2567}) + \lambda_6(e_{2348} - \mu_3 e_{1567})\\ &+ \sigma_2(e_{1346} + \mu_3 e_{2578}) + \sigma_6(e_{2346} - \mu_3 e_{1578}), \end{aligned}$$

subject to the second equation in (23). This is precisely the equation that allows us to write F as a sum of two simple forms $F = F_1 + \mu_3 F_2$, where

$$\begin{aligned} F_1 &= (e_1 - \sigma_6 e_6 - \lambda_6 e_8) \wedge (e_2 + \sigma_2 e_6 + \lambda_2 e_8) \wedge e_3 \wedge e_4,\\ F_2 &= e_5 \wedge (e_6 + \sigma_6 e_1 - \sigma_2 e_2) \wedge e_7 \wedge (e_8 + \lambda_6 e_1 - \lambda_2 e_2). \end{aligned}$$

Notice moreover that F_1 and F_2 are orthogonal.

If $\lambda_2 = 0$ and $\sigma_2 \neq 0$, F is given by

$$F = e_{1234} + \mu_3 e_{5678} + \sigma_2(e_{1346} + \mu_3 e_{2578}) + \sigma_6(e_{2346} - \mu_3 e_{1578}),$$

which can be written as a sum $F = F_1 + \mu_3 F_2$ of two simple forms

$$\begin{aligned} F_1 &= (e_1 - \sigma_6 e_6) \wedge (e_2 + \sigma_2 e_6) \wedge e_3 \wedge e_4,\\ F_2 &= e_5 \wedge (e_6 + \sigma_6 e_1 - \sigma_2 e_2) \wedge e_7 \wedge e_8, \end{aligned}$$

which moreover are orthogonal.

Finally, if $\lambda_2 = \sigma_2 = 0$, F is given by

$$F = e_{1234} + \mu_3 e_{5678} + \lambda_5(e_{2347} + \mu_3 e_{1568}) + \lambda_6(e_{2348} - \mu_3 e_{1567}) + \sigma_5(e_{2345} + \mu_3 e_{1678}) + \sigma_6(e_{2346} - \mu_3 e_{1578}),$$

which can be written as a sum of two orthogonal simple forms $F = F_1 + \mu_3 F_2$, where

$$F_1 = (e_1 - \sigma_5 e_5 - \sigma_6 e_6 - \lambda_5 e_7 - \lambda_6 e_8) \wedge e_2 \wedge e_3 \wedge e_4,$$
$$F_2 = (e_5 + \sigma_5 e_1) \wedge (e_6 + \sigma_6 e_1) \wedge (e_7 + \lambda_5 e_1) \wedge (e_8 + \lambda_6 e_1).$$

2.7. *Proof for $F \in \Lambda^4\mathbb{E}^7$*

Choose an orthonormal basis $\{e_1, e_2, \dots, e_7\}$ for which $\iota_{12}F = \alpha e_{34} + \beta e_{56}$, where ι_{12} means the contraction of F by e_{12}.

Suppose that α and β are generic. In this case, the equation $[\iota_{12}F, F] = 0$ says that the only terms in F which survive are those which are invariant under the maximal torus of SO(5), the group of rotations in the five-dimensional space spanned by $\{e_3, e_4, \dots, e_7\}$; that is,

$$F = \alpha e_{1234} + \beta e_{1256} + \gamma e_{3456}.$$

Now $[\iota_{23}F, F] = 0$ implies that $\alpha\beta = 0$, violating the condition that $\iota_{12}F$ be generic.

Non-generic rotations correspond to (conjugacy classes of) subalgebras of $\mathfrak{so}(5)$ with rank strictly less than that of $\mathfrak{so}(5)$:

(1) $\mathfrak{su}(2)$: $\alpha = \beta \neq 0$; and
(2) $\mathfrak{so}(2)$: $\beta = 0$ and $\alpha \neq 0$.

We now go down this list case by case.

2.7.1. $\mathfrak{su}(2)$

In this case $\iota_{12}F = \alpha(e_{34}+e_{56})$. This means that the most general solution of $[\iota_{12}F, F] = 0$ is given by

$$F = \alpha e_{1234} + \alpha e_{1256} + e_1 \wedge G_1 + e_2 \wedge G_2 + G_3,$$

where $G_1, G_2 \in \Lambda^3\mathbb{E}^5$ and $G_3 = \Lambda^4\mathbb{E}^5$, where $\mathbb{E}^5$ is spanned by $\{e_i\}_{3\leq i\leq 7}$, and where the G_i have zero weight with respect to this $\mathfrak{su}(2)$ algebra. A little group theory shows that G_1 and G_2 are linear combinations of the following eight 3-forms:

$$e_{347} + e_{567}, \quad e_{347} - e_{567}, \quad e_{357} + e_{467}, \quad e_{367} - e_{457},$$

whereas

$$G_3 = \mu e_{3456}.$$

This means that F takes the following form:

$$\begin{aligned}F = {}&\alpha e_{1234} + \alpha e_{1256} + \mu e_{3456} + \lambda_1(e_{1347} + e_{1567}) + \lambda_2(e_{1347} - e_{1567})\\ &+ \lambda_3(e_{1357} + e_{1467}) + \lambda_4(e_{1367} - e_{1457})\\ &+ \rho_1(e_{2347} + e_{2567}) + \rho_2(e_{2347} - e_{2567}) + \rho_3(e_{2357} + e_{2467})\\ &+ \rho_4(e_{2367} - e_{2457}).\end{aligned}$$

Rotating in the anti-self-dual (3456) plane allows us to set $\lambda_3 = \lambda_4 = 0$. Imposing, for example, the equation $[\iota_{23}F, F] = 0$ and $[\iota_{25}F, F] = 0$ tells us that $\lambda_1 = \lambda_2 = 0$. This allows us to rotate again in the anti-self-dual (3456) plane to set $\rho_3 = \rho_4 = 0$ and imposing $[\iota_{13}F, F] = 0$ and $[\iota_{15}F, F] = 0$ to find that $\rho_1 = \rho_2 = 0$. The remaining equations imply that $\alpha^2 = 0$ which is a contradiction.

2.7.2. $\mathfrak{so}(2)$

Finally, we consider the case where $\iota_{12}F = \alpha e_{34}$. The most general F has the form

$$F = \alpha e_{1234} + e_1 \wedge G_1 + e_2 \wedge G_2 + G_3,$$

where $G_1, G_2 \in \Lambda^3\mathbb{E}^5$ and $G_3 \in \Lambda^4\mathbb{E}^5$, where $\mathbb{E}^5$ is spanned by $\{e_i\}_{3\le i\le 7}$. Such an F will obey $[\iota_{12}F, F] = 0$ if and only if the G_i have zero weights under the $\mathfrak{so}(2)$ generated by $\iota_{12}F$. This means that each of G_1, G_2 is a linear combination of the four monomials

$$e_{345}, \quad e_{346}, \quad e_{347}, \quad e_{567}.$$

Using the freedom to conjugate by the SO(3) which acts in the (567) plane, we can write

$$G_3 = \mu e_{3456}.$$

So F is

$$F = \alpha e_{1234} + \mu e_{3456} + \lambda_1 e_{1345} + \lambda_2 e_{1346} + \lambda_3 e_{1567} + \sigma_1 e_{2345} + \sigma_2 e_{2346} + \sigma_3 e_{2567}.$$

Rotating in the (56) plane, we can set $\lambda_2 = 0$. Suppose that $\mu \neq 0$. In this case $[\iota_{36}F, F] = 0$ implies that $\lambda_3 = \sigma_3 = 0$. Next observe that $\iota_{34}F$ is a 2-form in $\mathbb{E}^4$ spanned by $\{e_1, e_2, e_5, e_6\}$. If $\iota_{34}F$ has rank 4 then it is the previous case which has led to a contradiction. If it has rank 2, then the statement is shown.

It remains to show the statement for $\mu = 0$. In this case, after performing a rotation in the (56) plane and setting $\lambda_2 = 0$, we have

$$F = \alpha e_{1234} + \lambda_1 e_{1345} + \lambda_3 e_{1567} + \sigma_1 e_{2345} + \sigma_2 e_{2346} + \sigma_3 e_{2567}.$$

One of the $[\iota_{13}F, F] = 0$ conditions implies that $\lambda_1\sigma_2 = 0$. If $\lambda_1 = 0$, using a rotation in the (56) plane, we can set $\sigma_2 = 0$ as well. The conditions $[\iota_{13}F, F] = 0$ and $[\iota_{23}F, F] = 0$ imply that $\lambda_3 = \sigma_3 = 0$. Thus

$$F = \alpha e_{1234} + \sigma_1 e_{2345} = (\alpha e_1 - \sigma_1 e_5) \wedge e_{234}$$

and it is simple. If instead $\sigma_2 = 0$, using a rotation in the (12) plane we can set $\lambda_1 = 0$. Then an analysis similar to the above yields that F is simple.

2.8. *Proof for* $F \in \Lambda^4\mathbb{E}^d$ *for* $d = 5, 6$

Choose an orthonormal basis in $\mathbb{E}^6\{e_1, e_2, \ldots, e_6\}$ for which $\iota_{12}F = \alpha e_{34} + \beta e_{56}$, where ι_{12} means the contraction of F by e_{12}.

Suppose that α and β are generic. In this case, the equation $[\iota_{12}F, F] = 0$ says that the only terms in F which survive are those which are invariant under the maximal torus of SO(4), the group of rotations in the five-dimensional space spanned by $\{e_3, e_4, \ldots, e_6\}$; that is,

$$F = \alpha e_{1234} + \beta e_{1256} + \gamma e_{3456}.$$

Now $[\iota_{23}F, F] = 0$ implies that $\alpha\beta = 0$, violating the condition that $\iota_{12}F$ be generic.

Non-generic rotations correspond to (conjugacy classes of) subalgebras of $\mathfrak{so}(4)$ with rank strictly less than that of $\mathfrak{so}(4)$:

(1) $\mathfrak{su}(2)$: $\alpha = \beta \neq 0$; and
(2) $\mathfrak{so}(2)$: $\beta = 0$ and $\alpha \neq 0$.

We now go down this list case by case.

2.8.1. $\mathfrak{su}(2)$

In this case $\iota_{12}F = \alpha(e_{34}+e_{56})$. This means that the most general solution of $[\iota_{12}F, F] = 0$ is given by

$$F = \alpha e_{1234} + \alpha e_{1256} + e_1 \wedge G_1 + e_2 \wedge G_2 + G_3,$$

where $G_1, G_2 \in \Lambda^3\mathbb{E}^4$ and $G_3 = \Lambda^4\mathbb{E}^4$, where $\mathbb{E}^4$ is spanned by $\{e_i\}_{3\leq i\leq 6}$, and where the G_i have zero weight with respect to this $\mathfrak{su}(2)$ algebra. A little group theory shows that $G_1 = G_2 = 0$ and

$$G_3 = \mu e_{3456}.$$

This means that F takes the following form:

$$F = \alpha e_{1234} + \alpha e_{1256} + \mu e_{3456}.$$

Imposing $[\iota_{23}F, F] = 0$ we find that $\alpha^2 = 0$ which is a contradiction.

2.8.2. $\mathfrak{so}(2)$

Finally, we consider the case where $\iota_{12}F = \alpha e_{34}$. The most general F has the form

$$F = \alpha e_{1234} + e_1 \wedge G_1 + e_2 \wedge G_2 + G_3,$$

where $G_1, G_2 \in \Lambda^3\mathbb{E}^4$ and $G_3 \in \Lambda^4\mathbb{E}^4$, where $\mathbb{E}^4$ is spanned by $\{e_i\}_{3\leq i\leq 6}$. Such an F will obey $[\iota_{12}F, F] = 0$ if and only if the G_i have zero weights under the $\mathfrak{so}(2)$ generated by $\iota_{12}F$. This means that each of G_1, G_2 is a linear combination of the two monomials e_{345} and e_{346}, whence

$$G_3 = \mu e_{3456},$$

and

$$F = \alpha e_{1234} + \mu e_{3456} + \lambda_1 e_{1345} + \lambda_2 e_{1346} + \sigma_1 e_{2345} + \sigma_2 e_{2346}.$$

Rotating in the (56) plane, we can set $\lambda_2 = 0$. Suppose that $\mu \neq 0$. Next observe that $\iota_{34}F$ is a 2-form in $\mathbb{E}^4$ spanned by $\{e_1, e_2, e_5, e_6\}$. If $\iota_{34}F$ has rank 4 then it is the previous case which has led to a contradiction. If it has rank 2, then the statement is shown.

It remains to show the statement for $d = 5$. In this case

$$F = \alpha e_{1234} + \beta e_{1534} + \gamma e_{2534}.$$

The 2-form $\iota_{34}F$ has rank 2 in $\mathbb{E}^3$ spanned by $\{e_1, e_2, e_3\}$ and the statement is shown.

2.9. Proof for $F \in \Lambda^5\mathbb{E}^{10}$

We shall not give the details of the proof of the conjecture in this case. This is because the proof follows closely that of $F \in \Lambda^5\mathbb{E}^{1,9}$ which will be given explicitly below. The only difference is certain signs in the various orthogonality relations that involve the "time" direction. The rest of the proof follows unchanged.

2.10. Proof for $F \in \Lambda^5\mathbb{E}^{1,9}$

Let us choose a pseudo-orthonormal basis $\{e_0, e_1, \dots, e_9\}$ with e_0 time-like in such a way that the 2-form $\iota_{012}F$ takes the form

$$\iota_{012}F = \alpha e_{34} + \beta e_{56} + \gamma e_{78}.$$

Depending on the values of α, β and γ we have the same cases as in the case of $d = 4$ treated in the previous section. The most general F can be written as

$$F = \alpha e_{01234} + \beta e_{01256} + \gamma e_{01278} + e_{12} \wedge G_0 + e_{02} \wedge G_1 + e_{01} \wedge G_2 + e_0 \wedge H_0 + e_1 \wedge H_1 + e_2 \wedge H_2 + K, \tag{24}$$

where $G_i \in \Lambda^3\mathbb{E}^7$, $H_i \in \Lambda^4\mathbb{E}^7$ and $K \in \Lambda^5\mathbb{E}^7$, where $\mathbb{E}^7$ is spanned by $\{e_i\}_{3\le i\le 9}$. For all values of α, β, γ, the 2-form $\iota_{012}F$ is an element in a fixed Cartan subalgebra of $\mathfrak{so}(6)$, and in solving $[\iota_{012}F, F] = 0$ we will be determining which G_i, H_i and K have zero weights with respect to this element. We will first decompose the relevant exterior powers of $\mathbb{E}^7$ in $\mathfrak{so}(6)$ representations. First of all, notice that $\mathbb{E}^7 = \mathbb{E}^6 \oplus \mathbb{R}$, where $\mathbb{E}^6$ is the vector representation of $\mathfrak{so}(6)$ and $\mathbb{R}$ is the span of e_9. This means that we can refine the above decomposition of F and notice that each G_i and each H_i will be written as follows:

$$G_i = L_i + M_i \wedge e_9, \quad \text{and} \quad H_i = N_i + P_i \wedge e_9,$$

where $M_i \in \Lambda^2\mathbb{E}^6$, $L_i, P_i \in \Lambda^3\mathbb{E}^6$ and $N_i \in \Lambda^4\mathbb{E}^6$. Since $\Lambda^4\mathbb{E}^6 \cong \Lambda^2\mathbb{E}^6$, we need only decompose $\Lambda^2\mathbb{E}^6$ and $\Lambda^3\mathbb{E}^6$. Clearly $\Lambda^2\mathbb{E}^6 \cong \mathfrak{so}(6)$ is nothing but the 15-dimensional adjoint representation with three zero weights corresponding to the Cartan subalgebra, whereas $\Lambda^3\mathbb{E}^6$ is a 20-dimensional irreducible representation having no zero weights with respect to $\mathfrak{so}(6)$; although of course it many have zero weights with respect to subalgebras of $\mathfrak{so}(6)$. Finally, let us mention that as we saw in the previous section, we will always be able to choose K to be a linear combination of the monomials e_{34569}, e_{34789}, e_{56789} by using the freedom to conjugate by the normaliser of the Cartan subalgebra in which $\iota_{012}F$ lies.

We have different cases to consider depending on the values of α, β and γ and as in the previous section we can label them according to the subalgebra of $\mathfrak{so}(6)$ in whose Cartan subalgebra they lie:

(1) $\mathfrak{so}(6)$: α, β and γ generic;
(2) $\mathfrak{su}(3)$: $\alpha+\beta+\gamma=0$ but all α, β, and γ non-zero;
(3) $\mathfrak{su}(2)\times\mathfrak{u}(1)$: $\alpha=\beta\neq\gamma$, but again all non-zero;
(4) $\mathfrak{u}(1)$ diagonal: $\alpha=\beta=\gamma\neq 0$;
(5) $\mathfrak{so}(4)$: $\gamma=0$ and $\alpha\neq\beta$ non-zero;
(6) $\mathfrak{su}(2)$: $\gamma=0$ and $\alpha=\beta\neq 0$; and
(7) $\mathfrak{so}(2)$: $\beta=\gamma=0$ and $\alpha\neq 0$.

We now go down this list case by case.

2.10.1. $\mathfrak{so}(6)$

The generic case is easy to discard. The most general F obeying $[\iota_{012}F, F]=0$ has 21 free parameters:

$$\begin{aligned}F ={}& \alpha e_{01234}+\beta e_{01256}+\gamma e_{01278}+\mu_1 e_{34569}+\mu_2 e_{34789}+\mu_3 e_{56789}+\lambda_1 e_{01349}\\ &+\lambda_2 e_{02349}+\lambda_3 e_{12349}+\lambda_4 e_{01569}+\lambda_5 e_{02569}+\lambda_6 e_{12569}+\lambda_7 e_{01789}+\lambda_8 e_{02789}\\ &+\lambda_9 e_{12789}+\sigma_1 e_{03456}+\sigma_2 e_{03478}+\sigma_3 e_{05678}\\ &+\sigma_4 e_{1345}+\sigma_5 e_{13478}+\sigma_6 e_{15678}+\sigma_7 e_{23456}+\sigma_8 e_{23478}+\sigma_9 e_{25678}.\end{aligned}$$

If we now consider the equation $[\iota_{013}F, F]=0$ we see that it is not satisfied unless either α or β are zero, violating the condition of genericity.

2.10.2. $\mathfrak{su}(3)$

As discussed above, the $\mathfrak{su}(3)$ zero weights in the representations $\Lambda^2\mathbb{E}^6$ and $\Lambda^3\mathbb{E}^6$ are linear combinations of the following forms:

$$e_{34},\quad e_{56},\quad e_{78},\quad \Omega_1,\quad \Omega_2,$$

where Ω_i are defined in Eq. (17). The most general F satisfying $[\iota_{012}F, F]=0$ is given by

$$\begin{aligned}F ={}& \alpha(e_{01234}-e_{01278})+\beta(e_{01256}-e_{01278})+\mu_1 e_{34569}+\mu_2 e_{34789}+\mu_3 e_{56789}\\ &+\lambda_1 e_{01349}+\lambda_2 e_{02349}+\lambda_3 e_{12349}+\lambda_4 e_{01569}+\lambda_5 e_{02569}+\lambda_6 e_{12569}+\lambda_7 e_{01789}\\ &+\lambda_8 e_{02789}+\lambda_9 e_{12789}+\sigma_1 e_{03456}+\sigma_2 e_{03478}+\sigma_3 e_{05678}+\sigma_4 e_{1345}+\sigma_5 e_{13478}\\ &+\sigma_6 e_{15678}+\sigma_7 e_{23456}+\sigma_8 e_{23478}+\sigma_9 e_{25678}+\rho_1 e_{01}\wedge\Omega_1+\rho_2 e_{02}\wedge\Omega_1\\ &+\rho_3 e_{12}\wedge\Omega_1+\rho_4 e_{01}\wedge\Omega_2+\rho_5 e_{02}\wedge\Omega_2+\rho_6 e_{12}\wedge\Omega_2-\tau_1 e_{09}\wedge\Omega_1\\ &-\tau_2 e_{19}\wedge\Omega_1-\tau_3 e_{29}\wedge\Omega_1-\tau_4 e_{09}\wedge\Omega_2-\tau_5 e_{19}\wedge\Omega_2-\tau_6 e_{29}\wedge\Omega_2.\end{aligned}$$

There are thus 33 free parameters, which we can reduce to 32 as was done in the previous section. Inspection of (some of) the remaining 30239 equations $[\iota_{ijk}F, F] = 0$ shows that α and β are constrained to obey $\alpha = \pm\beta$, violating the hypothesis of genericity.

2.10.3. $\mathfrak{su}(2) \times \mathfrak{u}(1)$

We now let $\alpha = \beta$, with the opposite case being related by an outer automorphism. As mentioned above $\Lambda^3\mathbb{E}^6$ has no zero weights, whereas those in $\Lambda^2\mathbb{E}^6$ are linear combinations of the following forms:

$$e_{34} + e_{56}, \quad e_{34} - e_{56}, \quad e_{35} + e_{46}, \quad e_{36} - e_{45}, \quad e_{78}.$$

The first and last are the generators of the Cartan subalgebra of $\mathfrak{su}(2) \times \mathfrak{u}(1)$ whereas the remaining three are the generators of the anti-self-dual $\mathfrak{su}(2) \subset \mathfrak{so}(4)$. Using the freedom to conjugate by the anti-self-dual $\mathfrak{su}(2)$ we will be able to eliminate two of the free parameters in the expression for F, which after this simplification takes the following form:

$$\begin{aligned} F = {}& \alpha(e_{01234} + e_{01256}) + \gamma e_{01278} + \mu_1 e_{34569} + \mu_2 e_{34789} + \mu_3 e_{56789} + \lambda_1 e_{01349} \\ &+ \lambda_2 e_{02349} + \lambda_3 e_{12349} + \lambda_4 e_{01569} + \lambda_5 e_{02569} + \lambda_6 e_{12569} + \lambda_7 e_{01789} + \lambda_8 e_{02789} \\ &+ \lambda_9 e_{12789} + \sigma_1 e_{03456} + \sigma_2 e_{03478} + \sigma_3 e_{05678} + \sigma_4 e_{1345} + \sigma_5 e_{13478} + \sigma_6 e_{15678} \\ &+ \sigma_7 e_{23456} + \sigma_8 e_{23478} + \sigma_9 e_{25678} + \rho_1(e_{01359} + e_{01469}) + \rho_2(e_{02359} + e_{02469}) \\ &+ \rho_3(e_{12359} + e_{12469}) + \rho_4(e_{01369} - e_{01459}) + \rho_5(e_{02369} - e_{02459}) \\ &+ \rho_6(e_{12369} - e_{12459}) + \tau_1(e_{04678} + e_{03578}) + \tau_2(e_{04578} - e_{03678}) \\ &+ \tau_3(e_{14678} + e_{13578}) + \tau_4(e_{14578} - e_{13678}) + \tau_5(e_{24678} + e_{23578}) \\ &+ \tau_6(e_{24578} - e_{23678}), \end{aligned}$$

which depends on 33 parameters. Inspection of the remaining equations immediately shows that $\alpha\gamma = 0$, violating genericity.

2.10.4. $\mathfrak{u}(1)$ *diagonal*

We now let $\alpha = \beta = \gamma$. As mentioned in the analogous case in the previous section, $\Lambda^3\mathbb{E}^6$ has no zero weights, whereas those in $\Lambda^2\mathbb{E}^6$ are linear combinations of the $\mathfrak{u}(3)$ generators ω_i:

$$\begin{aligned} &e_{35} + e_{46}, \quad e_{45} - e_{36}, \quad e_{37} + e_{48}, \quad e_{47} - e_{38}, \\ &e_{57} + e_{68}, \quad e_{67} - e_{58}, \quad e_{34}, \quad e_{56}, \quad e_{78}. \end{aligned}$$

We have the freedom to conjugate by the normaliser of this $\mathfrak{u}(1)$ in $\mathfrak{so}(6)$, which is precisely $\mathfrak{u}(3)$. This means that we can conjugate the $\mathfrak{u}(3)$ generators in the form K in (24) to a Cartan subalgebra of $\mathfrak{u}(3)$. In summary the most general F contains 57 parameters and can be written as

$$\begin{aligned} F = {}& \alpha(e_{01234} + e_{01256} + e_{01278}) + \mu_1 e_{34569} + \mu_2 e_{34789} + \mu_3 e_{56789} \\ &+ \sum_{i=1}^{9}(\lambda_i e_{01} + \lambda_{9+i} e_{02} + \lambda_{18+i} e_{12}) \wedge \omega_i + \sum_{i=1}^{9}(\sigma_i e_0 + \sigma_{9+i} e_1 + \sigma_{18+i} e_2) \wedge \star\omega_i, \end{aligned}$$

where $\star\omega_i \in \Lambda^4\mathbb{E}^6$ are the Hodge duals of the ω_i. Inspection of a few of the remaining equations shows that they are consistent only if $\alpha = 0$, which violates the hypothesis.

As in the eight-dimensional case treated in the previous section, there are no solutions when $\iota_{012}F$ has maximal rank, a fact which again lacks a simpler explanation.

2.10.5. $\mathfrak{so}(4)$

Let $\iota_{012}F = \alpha e_{34} + \beta e_{56}$ with α and β generic. The condition that $[\iota_{012}F, F] = 0$ means that F takes the form given by Eq. (24) where $G_i \in \Lambda^3\mathbb{E}^7$ are linear combinations of the six monomials

$$e_{347}, \quad e_{348}, \quad e_{349}, \quad e_{567}, \quad e_{568}, \quad e_{569},$$

where the $H_i \in \Lambda^4\mathbb{E}^7$ are linear combinations of their duals

$$e_{5689}, \quad e_{5679}, \quad e_{5678}, \quad e_{3489}, \quad e_{3479}, \quad e_{3478}.$$

The 5-form K is as usual a linear combination of the three monomials: e_{34569}, e_{34789}, e_{56789}. In summary, F is given by the following expression containing 39 free parameters:

$$\begin{aligned} F = {} & \alpha e_{01234} + \beta e_{01256} + \mu_1 e_{34569} + \mu_2 e_{34789} + \mu_3 e_{56789} + \lambda_1 e_{01347} + \lambda_2 e_{02347} \\ & + \lambda_3 e_{12347} + \lambda_4 e_{01348} + \lambda_5 e_{02348} + \lambda_6 e_{12348} + \lambda_7 e_{01349} + \lambda_8 e_{02349} \\ & + \lambda_9 e_{12349} + \sigma_1 e_{01567} + \sigma_2 e_{02567} + \sigma_3 e_{12567} + \sigma_4 e_{01568} + \sigma_5 e_{02568} \\ & + \sigma_6 e_{12568} + \sigma_7 e_{01569} + \sigma_8 e_{02569} + \sigma_9 e_{12569} + \rho_1 e_{03478} + \rho_2 e_{13478} + \rho_3 e_{23478} \\ & + \rho_4 e_{03479} + \rho_5 e_{13479} + \rho_6 e_{23479} + \rho_7 e_{03489} + \rho_8 e_{13489} + \rho_9 e_{23489} + \tau_1 e_{05678} \\ & + \tau_2 e_{15678} + \tau_3 e_{25678} + \tau_4 e_{05679} + \tau_5 e_{15679} + \tau_6 e_{25679} + \tau_7 e_{05689} \\ & + \tau_8 e_{15689} + \tau_9 e_{25689}. \end{aligned}$$

We can still rotate in the (12) and (78) planes and in this way set to zero two of the above parameters, say σ_3 and ρ_3, although we do not gain much from it. The equations $[\iota_{ijk}F, F] = 0$ have solutions for every α, β. Setting $\alpha = 1$ without loss of generality, we find that $\mu_1 = 0$ and that all the variables are given in terms of the λ_i which remain unconstrained:

$$\begin{aligned} &\tau_1 = \mu_3\lambda_9, && \sigma_1 = -\mu_3\rho_9, && \rho_1 = \lambda_1\lambda_5 - \lambda_2\lambda_4, \\ &\tau_2 = \mu_3\lambda_8, && \sigma_2 = \mu_3\rho_8, && \rho_2 = \lambda_1\lambda_6 - \lambda_3\lambda_4, \\ &\tau_3 = -\mu_3\lambda_7, && \sigma_3 = \mu_3\rho_7, && \rho_3 = \lambda_2\lambda_6 - \lambda_3\lambda_5, \\ &\tau_4 = -\mu_3\lambda_6, && \sigma_4 = \mu_3\rho_6, && \rho_4 = \lambda_1\lambda_8 - \lambda_2\lambda_7, \\ &\tau_5 = -\mu_3\lambda_5, && \sigma_5 = -\mu_3\rho_5, && \rho_5 = \lambda_1\lambda_9 - \lambda_3\lambda_7, \\ &\tau_6 = \mu_3\lambda_4, && \sigma_6 = -\mu_3\rho_4, && \rho_6 = \lambda_2\lambda_9 - \lambda_3\lambda_8, \\ &\tau_7 = \mu_3\lambda_3, && \sigma_7 = -\mu_3\rho_3, && \rho_7 = \lambda_4\lambda_8 - \lambda_5\lambda_7, \\ &\tau_8 = \mu_3\lambda_2, && \sigma_8 = \mu_3\rho_2, && \rho_8 = \lambda_4\lambda_9 - \lambda_6\lambda_7, \\ &\tau_9 = -\mu_3\lambda_1, && \sigma_9 = \mu_3\rho_1, && \rho_9 = \lambda_5\lambda_9 - \lambda_6\lambda_8, \end{aligned}$$

and

$$\mu_2 = \lambda_1\lambda_5\lambda_9 - \lambda_3\lambda_5\lambda_7 + \lambda_2\lambda_6\lambda_7 + \lambda_3\lambda_4\lambda_8 - \lambda_1\lambda_6\lambda_8 - \lambda_2\lambda_4\lambda_9,$$

subject to one equation

$$\beta = \mu_2\mu_3. \tag{25}$$

Remarkably (perhaps) these equations are precisely the ones that guarantee that F can be written as a sum of two simple forms

$$F = \theta_0 \wedge \theta_1 \wedge \theta_2 \wedge e_3 \wedge e_4 + \mu_3 e_5 \wedge e_6 \wedge \theta_7 \wedge \theta_8 \wedge \theta_9,$$

where

$$\begin{aligned}
\theta_0 &= e_0 + \lambda_3 e_7 + \lambda_6 e_8 + \lambda_9 e_9, & \theta_7 &= e_7 + \lambda_3 e_0 + \lambda_2 e_1 - \lambda_1 e_2,\\
\theta_1 &= e_1 - \lambda_2 e_7 - \lambda_5 e_8 - \lambda_8 e_9, & \theta_8 &= e_8 + \lambda_6 e_0 + \lambda_5 e_1 - \lambda_4 e_2,\\
\theta_2 &= e_2 + \lambda_1 e_7 + \lambda_4 e_8 + \lambda_7 e_9, & \theta_9 &= e_9 + \lambda_9 e_0 + \lambda_8 e_1 - \lambda_7 e_2.
\end{aligned}$$

Notice moreover that $\theta_i \perp \theta_j$ for $i = 0, 1, 2$ and $j = 7, 8, 9$, whence the conjecture holds.

2.10.6. $\mathfrak{su}(2)$

Let $\iota_{012}F = \alpha(e_{01234} + e_{01256})$, where we can put $\alpha = 1$ without loss of generality. The most general solution of $[\iota_{012}F, F] = 0$ takes the form (24) where K is as usual a linear combination of the three monomials $e_{34569}, e_{34789}, e_{56789}$, the G_i are linear combinations of the following 3-forms:

$$e_{34i} + e_{56i}, \quad e_{34i} - e_{56i}, \quad e_{35i} + e_{46i}, \quad e_{36i} - e_{45i}, \quad e_{789},$$

where $i = 7, 8, 9$, and the H_i are linear combinations of their duals. In total we have 81 free parameters:

$$\begin{aligned}
F = {}& e_{01234} + e_{01256} + \mu_1 e_{34569} + \mu_2 e_{34789} + \mu_3 e_{56789} + \lambda_1 e_{01347} + \lambda_2 e_{01348}\\
&+ \lambda_3 e_{01349} + \lambda_4 e_{01567} + \lambda_5 e_{01568} + \lambda_6 e_{01569} + \lambda_7 e_{01789} + \lambda_8 (e_{01357} + e_{01467})\\
&+ \lambda_9 (e_{01358} + e_{01468}) + \lambda_{10} (e_{01359} + e_{01469}) + \lambda_{11} (e_{01367} - e_{01457})\\
&+ \lambda_{12} (e_{01368} - e_{01458}) + \lambda_{13} (e_{01369} - e_{01459})\\
&+ \rho_1 e_{02347} + \rho_2 e_{02348} + \rho_3 e_{02349} + \rho_4 e_{02567} + \rho_5 e_{02568} + \rho_6 e_{02569} + \rho_7 e_{02789}\\
&+ \rho_8 (e_{02357} + e_{02467}) + \rho_9 (e_{02358} + e_{02468}) + \rho_{10} (e_{02359} + e_{02469})\\
&+ \rho_{11} (e_{02367} - e_{02457}) + \rho_{12} (e_{02368} - e_{02458}) + \rho_{13} (e_{02369} - e_{02459})\\
&+ \sigma_1 e_{12347} + \sigma_2 e_{12348} + \sigma_3 e_{12349} + \sigma_4 e_{12567} + \sigma_5 e_{12568} + \sigma_6 e_{12569} + \sigma_7 e_{12789}\\
&+ \sigma_8 (e_{12357} + e_{12467}) + \sigma_9 (e_{12358} + e_{12468}) + \sigma_{10} (e_{12359} + e_{12469})\\
&+ \sigma_{11} (e_{12367} - e_{12457}) + \sigma_{12} (e_{12368} - e_{12458}) + \sigma_{13} (e_{12369} - e_{12459})\\
&+ \eta_1 e_{03456} + \eta_2 e_{03478} + \eta_3 e_{03479} + \eta_4 e_{03489} + \eta_5 e_{05678} + \eta_6 e_{05679} + \eta_7 e_{05689}\\
&+ \eta_8 (e_{03578} + e_{04678}) + \eta_9 (e_{03579} + e_{04679}) + \eta_{10} (e_{03589} + e_{04689})\\
&+ \eta_{11} (e_{03678} - e_{04578}) + \eta_{12} (e_{03679} - e_{04579}) + \eta_{13} (e_{03689} - e_{04589})\\
&+ \phi_1 e_{13456} + \phi_2 e_{13478} + \phi_3 e_{13479} + \phi_4 e_{13489} + \phi_5 e_{15678} + \phi_6 e_{15679} + \phi_7 e_{15689}\\
&+ \phi_8 (e_{13578} + e_{14678}) + \phi_9 (e_{13579} + e_{14679}) + \phi_{10} (e_{13589} + e_{14689})\\
&+ \phi_{11} (e_{13678} - e_{14578}) + \phi_{12} (e_{13679} - e_{14579}) + \phi_{13} (e_{13689} - e_{14589})
\end{aligned}$$

$$+\tau_1 e_{23456} + \tau_2 e_{23478} + \tau_3 e_{23479} + \tau_4 e_{23489} + \tau_5 e_{25678} + \tau_6 e_{25679} + \tau_7 e_{25689}$$
$$+\tau_8(e_{23578} + e_{24678}) + \tau_9(e_{23579} + e_{24679}) + \tau_{10}(e_{23589} + e_{24689})$$
$$+\tau_{11}(e_{23678} - e_{24578}) + \tau_{12}(e_{23679} - e_{24579}) + \tau_{13}(e_{23689} - e_{24589}).$$

We notice first of all that the equations $[\iota_{ijk}F, F] = 0$ imply that $\lambda_7 = \rho_7 = \sigma_7 = 0$ and after close inspection of the equations one can see that there are no solutions unless $\mu_1 = 0$, which we will assume from now on.

One then must distinguish between two cases, depending on whether or not μ_2 equals μ_3. Let us first of all consider the generic situation $\mu_2 \neq \mu_3$. One immediately sees that the following coefficients vanish: $\lambda_i = \rho_i = \sigma_i = \eta_i = \tau_i = \phi_i = 0$ for $i \geq 8$, leaving F in the following form:

$$F = e_{34} \wedge G_1 + e_{56} \wedge G_2,$$

where

$$\begin{aligned} G_1 = {} & e_{012} + \mu_2 e_{789} + \lambda_1 e_{017} + \lambda_2 e_{018} + \lambda_3 e_{019} + \rho_1 e_{027} + \rho_2 e_{028} + \rho_3 e_{029} \\ & + \sigma_1 e_{127} + \sigma_2 e_{128} + \sigma_3 e_{129} + \eta_2 e_{078} + \eta_3 e_{079} + \eta_4 e_{089} + \phi_2 e_{178} + \phi_3 e_{179} \\ & + \phi_4 e_{189} + \tau_2 e_{278} + \tau_3 e_{279} + \tau_4 e_{289} \end{aligned}$$

and

$$\begin{aligned} G_2 = {} & e_{012} + \mu_3 e_{789} + \lambda_4 e_{017} + \lambda_5 e_{018} + \lambda_6 e_{019} + \rho_4 e_{027} + \rho_5 e_{028} + \rho_6 e_{029} \\ & + \sigma_4 e_{127} + \sigma_5 e_{128} + \sigma_6 e_{129} + \eta_5 e_{078} + \eta_6 e_{079} + \eta_7 e_{089} + \phi_5 e_{178} + \phi_6 e_{179} \\ & + \phi_7 e_{189} + \tau_5 e_{278} + \tau_6 e_{279} + \tau_7 e_{289}. \end{aligned}$$

Some of the remaining equations express the η's, ϕ's and τ's in terms of the λ's, ρ's and σ's:

$$\begin{aligned} &\eta_2 = \mu_2\sigma_6, && \tau_2 = -\mu_2\lambda_6, && \phi_2 = \mu_2\rho_6, \\ &\eta_3 = -\mu_2\sigma_5, && \tau_3 = \mu_2\lambda_5, && \phi_3 = -\mu_2\rho_5, \\ &\eta_4 = \mu_2\sigma_4, && \tau_4 = -\mu_2\lambda_4, && \phi_4 = \mu_2\rho_4, \\ &\eta_5 = \mu_3\sigma_3, && \tau_5 = -\mu_3\lambda_3, && \phi_5 = \mu_3\rho_3, \\ &\eta_6 = -\mu_3\sigma_2, && \tau_6 = \mu_3\lambda_2, && \phi_6 = -\mu_3\rho_2, \\ &\eta_7 = \mu_3\sigma_1, && \tau_7 = -\mu_3\lambda_1, && \phi_7 = \mu_3\rho_1, \end{aligned}$$

whereas others in turn relate λ_i, ρ_i and σ_i for $i = 4, 5, 6$ to λ_j, ρ_j and σ_j for $j = 1, 2, 3$:

$$\begin{aligned} &\lambda_4 = \mu_3(\rho_3\sigma_2 - \rho_2\sigma_3), && \rho_4 = \mu_3(\lambda_2\sigma_3 - \lambda_3\sigma_2), && \sigma_4 = \mu_3(\lambda_2\rho_3 - \lambda_3\rho_2), \\ &\lambda_5 = \mu_3(\rho_1\sigma_3 - \rho_3\sigma_1), && \rho_5 = \mu_3(\lambda_3\sigma_1 - \lambda_1\sigma_3), && \sigma_5 = \mu_3(\lambda_3\rho_1 - \lambda_1\rho_3), \\ &\lambda_6 = \mu_3(\rho_2\sigma_1 - \rho_1\sigma_2), && \rho_6 = \mu_3(\lambda_1\sigma_2 - \lambda_2\sigma_1), && \sigma_6 = \mu_3(\lambda_1\rho_2 - \lambda_2\rho_1). \end{aligned}$$

The remaining independent variables are subject to two final equations:

$$\mu_2 = \sum_{\pi \in \mathfrak{S}_3} (-1)^{|\pi|} \lambda_{\pi(1)}\rho_{\pi(2)}\sigma_{\pi(3)} \quad \text{and} \quad \mu_2\mu_3 = 1, \tag{26}$$

where the sum in the first equation is over the permutations of three letters and weighted by the sign of the permutation. These equations guarantee that G_1 and G_2 are simple forms:

$$G_1 = \theta_0 \wedge \theta_1 \wedge \theta_2 \quad \text{and} \quad G_2 = \mu_3 \theta_7 \wedge \theta_8 \wedge \theta_9,$$

where

$$\begin{aligned}
&\theta_0 = e_0 + \sigma_1 e_7 + \sigma_2 e_8 + \sigma_3 e_9, \qquad \theta_1 = e_1 - \rho_1 e_7 - \rho_2 e_8 - \rho_3 e_9,\\
&\theta_2 = e_2 + \lambda_1 e_7 + \lambda_2 e_8 + \lambda_3 e_9,\\
&\theta_7 = e_7 + \sigma_1 e_0 + \rho_1 e_1 - \lambda_1 e_2, \qquad \theta_8 = e_8 + \sigma_2 e_0 + \rho_2 e_1 - \lambda_2 e_2,\\
&\theta_9 = e_9 + \sigma_3 e_0 + \rho_3 e_1 - \lambda_3 e_2.
\end{aligned}$$

If we define $\theta_i = e_i$ for $i = 3, 4, 5, 6$ then we see that the θ_i are mutually orthogonal and hence that

$$F = \theta_0 \wedge \theta_1 \wedge \theta_2 \wedge \theta_3 \wedge \theta_4 + \mu_3 \theta_5 \wedge \theta_6 \wedge \theta_7 \wedge \theta_8 \wedge \theta_9$$

is a sum of two orthogonal simple forms.

Finally we consider the case $\mu_2 = \mu_3$ which has no solution unless $\mu_2^2 = 1$. As in the case of 4-forms in eight dimensions treated in the previous section, we will show that we can choose a frame where the coefficients λ_i, ρ_i and σ_i vanish for $i \geq 8$, thus reducing this case to the generic case treated immediately above.

Some of the equations $[\iota_{ijk}F, F] = 0$ express the η's, τ's and ϕ's in terms of the λ's, ρ's and σ's, leaving F in the following form

$$\begin{aligned}
F = {}& e_{01234} + e_{01256} + \mu_2(e_{34789} + e_{56789}) + \lambda_1(e_{01347} - \mu_2 e_{25689})\\
&+ \lambda_2(e_{01348} + \mu_2 e_{25679}) + \lambda_3(e_{01349} - \mu_2 e_{25678}) + \lambda_4(e_{01567} - \mu_2 e_{23489})\\
&+ \lambda_5(e_{01568} + \mu_2 e_{23479}) + \lambda_6(e_{01569} - \mu_2 e_{23478})\\
&+ \lambda_8(e_{01357} + e_{01467} + \mu_2 e_{23589} + \mu_2 e_{24689})\\
&+ \lambda_9(e_{01358} + e_{01468} - \mu_2 e_{23579} - \mu_2 e_{24679})\\
&+ \lambda_{10}(e_{01359} + e_{01469} + \mu_2 e_{23578} + \mu_2 e_{24678})\\
&+ \lambda_{11}(e_{01367} - e_{01457} + \mu_2 e_{23689} - \mu_2 e_{24589})\\
&+ \lambda_{12}(e_{01368} - e_{01458} - \mu_2 e_{23679} + \mu_2 e_{24579})\\
&+ \lambda_{13}(e_{01369} - e_{01459} + \mu_2 e_{23678} - \mu_2 e_{24578})\\
&+ \rho_1(e_{02347} + \mu_2 e_{15689}) + \rho_2(e_{02348} - \mu_2 e_{15679}) + \rho_3(e_{02349} + \mu_2 e_{15678})\\
&+ \rho_4(e_{02567} + \mu_2 e_{13489}) + \rho_5(e_{02568} - \mu_2 e_{13479}) + \rho_6(e_{02569} + \mu_2 e_{13478})\\
&+ \rho_8(e_{02357} + e_{02467} - \mu_2 e_{13589} - \mu_2 e_{14689})\\
&+ \rho_9(e_{02358} + e_{02468} + \mu_2 e_{13579} + \mu_2 e_{14679})\\
&+ \rho_{10}(e_{02359} + e_{02469} - \mu_2 e_{13578} - \mu_2 e_{14678})\\
&+ \rho_{11}(e_{02367} - e_{02457} - \mu_2 e_{13689} + \mu_2 e_{14589})\\
&+ \rho_{12}(e_{02368} - e_{02458} + \mu_2 e_{13679} - \mu_2 e_{14579})
\end{aligned}$$

$$
\begin{aligned}
&+\rho_{13}(e_{02369} - e_{02459} - \mu_2 e_{13678} + \mu_2 e_{14578}) + \sigma_1(e_{12347} + \mu_2 e_{05689}) \\
&+\sigma_2(e_{12348} - \mu_2 e_{05679}) + \sigma_3(e_{12349} + \mu_2 e_{05678}) + \sigma_4(e_{12567} + \mu_2 e_{03489}) \\
&+\sigma_5(e_{12568} - \mu_2 e_{03479}) + \sigma_6(e_{12569} + \mu_2 e_{03478}) \\
&+\sigma_8(e_{12357} + e_{12467} - \mu_2 e_{03589} - \mu_2 e_{04689}) \\
&+\sigma_9(e_{12358} + e_{12468} + \mu_2 e_{03579} + \mu_2 e_{04679}) \\
&+\sigma_{10}(e_{12359} + e_{12469} - \mu_2 e_{03578} - \mu_2 e_{04678}) \\
&+\sigma_{11}(e_{12367} - e_{12457} - \mu_2 e_{03689} + \mu_2 e_{04589}) \\
&+\sigma_{12}(e_{12368} - e_{12458} + \mu_2 e_{03679} - \mu_2 e_{04579}) \\
&+\sigma_{13}(e_{12369} - e_{12459} - \mu_2 e_{03678} + \mu_2 e_{04578}).
\end{aligned}
$$

Let us define the following (anti)self-dual 3-forms in the (012789) plane:

$$
\begin{aligned}
\omega_0^\pm &= e_{012} \pm \mu_2 e_{789}, & \omega_5^\pm &= e_{028} \mp \mu_2 e_{179}, \\
\omega_1^\pm &= e_{017} \mp \mu_2 e_{289}, & \omega_6^\pm &= e_{029} \pm \mu_2 e_{178}, \\
\omega_2^\pm &= e_{018} \pm \mu_2 e_{279}, & \omega_7^\pm &= e_{127} \pm \mu_2 e_{089}, \\
\omega_3^\pm &= e_{019} \mp \mu_2 e_{278}, & \omega_8^\pm &= e_{128} \mp \mu_2 e_{079}, \\
\omega_4^\pm &= e_{027} \pm \mu_2 e_{189}, & \omega_9^\pm &= e_{129} \pm \mu_2 e_{078},
\end{aligned}
$$

and the following (anti)self-dual 2-forms in the (3456) plane:

$$
\Theta_1^\pm = e_{34} \pm e_{56}, \qquad \Theta_2^\pm = e_{35} \mp e_{46}, \qquad \Theta_3^\pm = e_{36} \pm e_{45},
$$

in terms of which we can rewrite F in a more transparent form:

$$
F = \Theta_1^+ \wedge \left(\omega_0^+ + \sum_{i=1}^{9} \nu_i^+ \omega_i^+ \right) + \sum_{i=1}^{9} \omega_i^- \Psi_i^-,
$$

where the Ψ_i^- are defined by

$$
\begin{aligned}
\Psi_1^- &= \nu_1^- \Theta_1^- + \lambda_8 \Theta_2^- + \lambda_{11} \Theta_3^-, & \Psi_2^- &= \nu_2^- \Theta_1^- + \lambda_9 \Theta_2^- + \lambda_{12} \Theta_3^-, \\
\Psi_3^- &= \nu_3^- \Theta_1^- + \lambda_{10} \Theta_2^- + \lambda_{13} \Theta_3^-, & \Psi_4^- &= \nu_4^- \Theta_1^- + \rho_8 \Theta_2^- + \rho_{11} \Theta_3^-, \\
\Psi_5^- &= \nu_5^- \Theta_1^- + \rho_9 \Theta_2^- + \rho_{12} \Theta_3^-, & \Psi_6^- &= \nu_6^- \Theta_1^- + \rho_{10} \Theta_2^- + \rho_{13} \Theta_3^-, \\
\Psi_7^- &= \nu_7^- \Theta_1^- + \sigma_8 \Theta_2^- + \sigma_{11} \Theta_3^-, & \Psi_8^- &= \nu_8^- \Theta_1^- + \sigma_9 \Theta_2^- + \sigma_{12} \Theta_3^-, \\
\Psi_9^- &= \nu_9^- \Theta_1^- + \sigma_{10} \Theta_2^- + \sigma_{13} \Theta_3^-,
\end{aligned}
$$

and where we have introduced the following variables

$$
\begin{aligned}
\nu_1^\pm &= \tfrac{1}{2}(\lambda_1 \pm \lambda_4), & \nu_4^\pm &= \tfrac{1}{2}(\rho_1 \pm \rho_4), & \nu_7^\pm &= \tfrac{1}{2}(\sigma_1 \pm \sigma_4), \\
\nu_2^\pm &= \tfrac{1}{2}(\lambda_2 \pm \lambda_5), & \nu_5^\pm &= \tfrac{1}{2}(\rho_2 \pm \rho_5), & \nu_8^\pm &= \tfrac{1}{2}(\sigma_2 \pm \sigma_5), \\
\nu_3^\pm &= \tfrac{1}{2}(\lambda_3 \pm \lambda_6), & \nu_6^\pm &= \tfrac{1}{2}(\rho_3 \pm \rho_6), & \nu_9^\pm &= \tfrac{1}{2}(\sigma_3 \pm \sigma_6).
\end{aligned}
$$

Some of the remaining equations $[\iota_{ijk} F, F] = 0$ now say that the nine anti-self-dual 2-forms Ψ_i^- are collinear. This means that by an anti-self-dual rotation in the (3456) plane we can set $\lambda_i = \rho_i = \sigma_i = 0$ for $i \geq 8$. We have therefore managed to reduce this case to the generic case ($\mu_2 \neq \mu_3$) except that now $\mu_2 = \mu_3$; but this was shown above to verify the conjecture.

2.10.7. $\mathfrak{so}(2)$

Let $\iota_{012}F = \alpha e_{01234}$, where we can put $\alpha = 1$ without loss of generality. The most general solution of $[\iota_{012}F, F] = 0$ takes the form (24) where the K is as usual a linear combination of the three monomials $e_{34569}, e_{34789}, e_{56789}$, the G_i are linear combinations of the following 3-forms:

$$\begin{array}{lllll} e_{345}, & e_{346}, & e_{347}, & e_{348}, & e_{349}, \\ e_{567}, & e_{568}, & e_{569}, & e_{578}, & e_{579}, \\ e_{589}, & e_{678}, & e_{679}, & e_{689}, & e_{789}, \end{array}$$

and the H_i are linear combinations of their duals. The most general solution to $[\iota_{012}F, F] = 0$ has 93 free parameters:

$$\begin{aligned} F = {} & e_{01234} + \mu_1 e_{34569} + \mu_2 e_{34789} + \mu_3 e_{56789} + \lambda_1 e_{01345} + \lambda_2 e_{01346} + \lambda_3 e_{01347} \\ & + \lambda_4 e_{01348} + \lambda_5 e_{01349} + \lambda_6 e_{01567} + \lambda_7 e_{01568} + \lambda_8 e_{01569} + \lambda_9 e_{01578} \\ & + \lambda_{10} e_{01579} + \lambda_{11} e_{01589} + \lambda_{12} e_{01678} + \lambda_{13} e_{01679} + \lambda_{14} e_{01689} + \lambda_{15} e_{01789} \\ & + \sigma_1 e_{02345} + \sigma_2 e_{02346} + \sigma_3 e_{02347} + \sigma_4 e_{02348} + \sigma_5 e_{02349} + \sigma_6 e_{02567} + \sigma_7 e_{02568} \\ & + \sigma_8 e_{02569} + \sigma_9 e_{02578} + \sigma_{10} e_{02579} + \sigma_{11} e_{02589} + \sigma_{12} e_{02678} + \sigma_{13} e_{02679} \\ & + \sigma_{14} e_{02689} + \sigma_{15} e_{02789} + \rho_1 e_{12345} + \rho_2 e_{12346} + \rho_3 e_{12347} + \rho_4 e_{12348} \\ & + \rho_5 e_{12349} + \rho_6 e_{12567} + \rho_7 e_{12568} + \rho_8 e_{12569} + \rho_9 e_{12578} + \rho_{10} e_{12579} \\ & + \rho_{11} e_{12589} + \rho_{12} e_{12678} + \rho_{13} e_{12679} + \rho_{14} e_{12689} + \rho_{15} e_{12789} + \tau_1 e_{03456} \\ & + \tau_2 e_{03457} + \tau_3 e_{03458} + \tau_4 e_{03459} + \tau_5 e_{03467} + \tau_6 e_{03468} + \tau_7 e_{03469} + \tau_8 e_{03478} \\ & + \tau_9 e_{03479} + \tau_{10} e_{03489} + \tau_{11} e_{05678} + \tau_{12} e_{05679} + \tau_{13} e_{05689} + \tau_{14} e_{05789} \\ & + \tau_{15} e_{06789} + \phi_1 e_{13456} + \phi_2 e_{13457} + \phi_3 e_{13458} + \phi_4 e_{13459} + \phi_5 e_{13467} \\ & + \phi_6 e_{13468} + \phi_7 e_{13469} + \phi_8 e_{13478} + \phi_9 e_{13479} + \phi_{10} e_{13489} + \phi_{11} e_{15678} \\ & + \phi_{12} e_{15679} + \phi_{13} e_{15689} + \phi_{14} e_{15789} + \phi_{15} e_{16789} + \eta_1 e_{23456} + \eta_2 e_{23457} \\ & + \eta_3 e_{23458} + \eta_4 e_{23459} + \eta_5 e_{23467} + \eta_6 e_{23468} + \eta_7 e_{23469} + \eta_8 e_{23478} + \eta_9 e_{23479} \\ & + \eta_{10} e_{23489} + \eta_{11} e_{25678} + \eta_{12} e_{25679} + \eta_{13} e_{25689} + \eta_{14} e_{25789} + \eta_{15} e_{26789}. \end{aligned}$$

First we consider the case where $\mu_1 \neq 0$. This means that many of the parameters must vanish: $\mu_2 = \mu_3 = 0$, $\eta_i = \phi_i = \tau_i = 0$ for $i \neq 1, 4, 7$ and $\lambda_j = \rho_j = \sigma_j = 0$ for $j \neq 1, 2, 5$. The resulting F can be written as $F = e_{34} \wedge G$, where

$$\begin{aligned} G = {} & e_{012} + \mu_1 e_{569} + \lambda_1 e_{015} + \lambda_2 e_{016} + \lambda_5 e_{019} + \sigma_1 e_{025} + \sigma_2 e_{026} + \sigma_5 e_{029} \\ & + \rho_1 e_{125} + \rho_2 e_{126} + \rho_5 e_{129} + \tau_1 e_{056} + \tau_4 e_{059} + \tau_7 e_{069} + \phi_1 e_{156} + \phi_4 e_{159} \\ & + \phi_7 e_{169} + \eta_1 e_{256} + \eta_4 e_{259} + \eta_7 e_{269}, \end{aligned}$$

where

$$\begin{aligned}
&\tau_1 = \lambda_1\sigma_2 - \lambda_2\sigma_1, \quad \phi_1 = \lambda_1\rho_2 - \lambda_2\rho_1, \quad \eta_1 = \sigma_1\rho_2 - \sigma_2\rho_1,\\
&\tau_4 = \lambda_1\sigma_5 - \lambda_5\sigma_1, \quad \phi_4 = \lambda_1\rho_5 - \lambda_5\rho_1, \quad \eta_4 = \sigma_1\rho_5 - \sigma_5\rho_1,\\
&\tau_7 = \lambda_2\sigma_5 - \lambda_5\sigma_2, \quad \phi_7 = \lambda_2\rho_5 - \lambda_5\rho_2, \quad \eta_7 = \sigma_2\rho_5 - \sigma_5\rho_2,
\end{aligned}$$

and subject to the equation

$$\mu_1 = \lambda_5\rho_2\sigma_1 - \lambda_2\rho_5\sigma_1 - \lambda_5\rho_1\sigma_2 + \lambda_1\rho_5\sigma_2 + \lambda_2\rho_1\sigma_5 - \lambda_1\rho_2\sigma_5, \tag{27}$$

which implies that G (and hence F) is simple:

$$\begin{aligned}
G = {}& (e_0 + \rho_1 e_5 + \rho_2 e_6 + \rho_5 e_9) \wedge (e_1 - \sigma_1 e_5 - \sigma_2 e_6 - \sigma_5 e_9)\\
&\wedge (e_2 + \lambda_1 e_5 + \lambda_2 e_6 + \lambda_5 e_9).
\end{aligned}$$

Let us assume from now on that $\mu_1 = 0$. If $\mu_2 \neq 0$ then the same conclusion as above obtains and F is simple. Details are the same up to a permutation of the orthonormal basis. We therefore assume that $\mu_2 = 0$. If $\mu_3 = 0$ then the following coefficients vanish: $\eta_i = \phi_i = \tau_i = 0$ for $i \geq 11$ and $\lambda_j = \rho_j = \sigma_j = 0$ for $j \geq 6$, resulting in $F = e_{34} \wedge G$, with

$$\begin{aligned}
G = {}& e_{012} + \lambda_1 e_{015} + \lambda_2 e_{016} + \lambda_3 e_{017} + \lambda_4 e_{018} + \lambda_5 e_{019} + \sigma_1 e_{025} + \sigma_2 e_{026}\\
&+ \sigma_3 e_{027} + \sigma_4 e_{028} + \sigma_5 e_{029} + \rho_1 e_{125} + \rho_2 e_{126} + \rho_3 e_{127} + \rho_4 e_{128} + \rho_5 e_{129}\\
&+ \tau_1 e_{056} + \tau_2 e_{057} + \tau_3 e_{058} + \tau_4 e_{059} + \tau_5 e_{067} + \tau_6 e_{068} + \tau_7 e_{069} + \tau_8 e_{078}\\
&+ \tau_9 e_{079} + \tau_{10} e_{089} + \phi_1 e_{156} + \phi_2 e_{157} + \phi_3 e_{158} + \phi_4 e_{159} + \phi_5 e_{167} + \phi_6 e_{168}\\
&+ \phi_7 e_{169} + \phi_8 e_{178} + \phi_9 e_{179} + \phi_{10} e_{189} + \eta_1 e_{256} + \eta_2 e_{257} + \eta_3 e_{258} + \eta_4 e_{259}\\
&+ \eta_5 e_{267} + \eta_6 e_{268} + \eta_7 e_{269} + \eta_8 e_{278} + \eta_9 e_{279} + \eta_{10} e_{289},
\end{aligned}$$

where

$$\begin{aligned}
&\phi_1 = \lambda_1\rho_2 - \lambda_2\rho_1, && \eta_1 = \sigma_1\rho_2 - \sigma_2\rho_1, && \tau_1 = \lambda_1\sigma_2 - \lambda_2\sigma_1,\\
&\phi_2 = \lambda_1\rho_3 - \lambda_3\rho_1, && \eta_2 = \sigma_1\rho_3 - \sigma_3\rho_1, && \tau_2 = \lambda_1\sigma_3 - \lambda_3\sigma_1,\\
&\phi_3 = \lambda_1\rho_4 - \lambda_4\rho_1, && \eta_3 = \sigma_1\rho_4 - \sigma_4\rho_1, && \tau_3 = \lambda_1\sigma_4 - \lambda_4\sigma_1,\\
&\phi_4 = \lambda_1\rho_5 - \lambda_5\rho_1, && \eta_4 = \sigma_1\rho_5 - \sigma_5\rho_1, && \tau_4 = \lambda_1\sigma_5 - \lambda_5\sigma_1,\\
&\phi_5 = \lambda_2\rho_3 - \lambda_3\rho_2, && \eta_5 = \sigma_2\rho_3 - \sigma_3\rho_2, && \tau_5 = \lambda_2\sigma_3 - \lambda_3\sigma_2,\\
&\phi_6 = \lambda_2\rho_4 - \lambda_4\rho_2, && \eta_6 = \sigma_2\rho_4 - \sigma_4\rho_2, && \tau_6 = \lambda_2\sigma_4 - \lambda_4\sigma_2,\\
&\phi_7 = \lambda_2\rho_5 - \lambda_5\rho_2, && \eta_7 = \sigma_2\rho_5 - \sigma_5\rho_2, && \tau_7 = \lambda_2\sigma_5 - \lambda_5\sigma_2,\\
&\phi_8 = \lambda_3\rho_4 - \lambda_4\rho_3, && \eta_8 = \sigma_3\rho_4 - \sigma_4\rho_3, && \tau_8 = \lambda_3\sigma_4 - \lambda_4\sigma_3,\\
&\phi_9 = \lambda_3\rho_5 - \lambda_5\rho_3, && \eta_9 = \sigma_3\rho_5 - \sigma_5\rho_3, && \tau_9 = \lambda_3\sigma_5 - \lambda_5\sigma_3,\\
&\phi_{10} = \lambda_4\rho_5 - \lambda_5\rho_4, && \eta_{10} = \sigma_4\rho_5 - \sigma_5\rho_4, && \tau_{10} = \lambda_4\sigma_5 - \lambda_5\sigma_4,
\end{aligned} \tag{28}$$

subject to the following 10 equations:

$$\sum_{\pi \in \mathfrak{S}_3} (-1)^{|\pi|} \lambda_{\pi(i)} \rho_{\pi(j)} \sigma_{\pi(k)} = 0 \tag{29}$$

for $1 \leq i < j < k \leq 5$, where the sum is over the permutations of three letters and weighted by the sign of the permutation. These equations are precisely the ones which guarantee that G (and hence F) is actually a simple form $G = \theta_0 \wedge \theta_1 \wedge \theta_2$, with

$$\theta_0 = e_0 + \rho_1 e_5 + \rho_2 e_6 + \rho_3 e_7 + \rho_4 e_8 + \rho_5 e_9,$$

$$\theta_1 = e_1 - \sigma_1 e_5 - \sigma_2 e_6 - \sigma_3 e_7 - \sigma_4 e_8 - \sigma_5 e_9,$$

$$\theta_2 = e_2 + \lambda_1 e_5 + \lambda_2 e_6 + \lambda_3 e_7 + \lambda_4 e_8 + \lambda_5 e_9.$$

Finally, if $\mu_3 \neq 0$ all that happens is that we find that the coefficients which vanish when $\mu_3 = 0$ are given in terms of those which do not by the following equations:

$$\begin{aligned}
&\eta_{15} = -\mu_3\lambda_1, && \phi_{15} = \mu_3\sigma_1, && \tau_{15} = \mu_3\rho_1,\\
&\eta_{14} = \mu_3\lambda_2, && \phi_{14} = -\mu_3\sigma_2, && \tau_{14} = -\mu_3\rho_2,\\
&\eta_{13} = -\mu_3\lambda_3, && \phi_{13} = \mu_3\sigma_3, && \tau_{13} = \mu_3\rho_3,\\
&\eta_{12} = \mu_3\lambda_4, && \phi_{12} = -\mu_3\sigma_4, && \tau_{12} = -\mu_3\rho_4,\\
&\eta_{11} = -\mu_3\lambda_5, && \phi_{11} = \mu_3\sigma_5, && \tau_{11} = \mu_3\rho_5,
\end{aligned}$$

and

$$\begin{aligned}
&\lambda_{15} = -\mu_3\eta_1, && \rho_{15} = -\mu_3\tau_1, && \sigma_{15} = \mu_3\phi_1,\\
&\lambda_{14} = \mu_3\eta_2, && \rho_{14} = -\mu_3\tau_2, && \sigma_{14} = -\mu_3\phi_2,\\
&\lambda_{13} = -\mu_3\eta_3, && \rho_{13} = \mu_3\tau_3, && \sigma_{13} = \mu_3\phi_3,\\
&\lambda_{12} = \mu_3\eta_4, && \rho_{12} = -\mu_3\tau_4, && \sigma_{12} = -\mu_3\phi_4,\\
&\lambda_{11} = -\mu_3\eta_5, && \rho_{11} = \mu_3\tau_5, && \sigma_{11} = \mu_3\phi_5,\\
&\lambda_{10} = \mu_3\eta_6, && \rho_{10} = -\mu_3\tau_6, && \sigma_{10} = -\mu_3\phi_6,\\
&\lambda_9 = -\mu_3\eta_7, && \rho_9 = \mu_3\tau_7, && \sigma_9 = \mu_3\phi_7,\\
&\lambda_8 = -\mu_3\eta_8, && \rho_8 = \mu_3\tau_8, && \sigma_8 = \mu_3\phi_8,\\
&\lambda_7 = \mu_3\eta_9, && \rho_7 = -\mu_3\tau_9, && \sigma_7 = -\mu_3\phi_9,\\
&\lambda_6 = -\mu_3\eta_{10}, && \rho_6 = \mu_3\tau_{10}, && \sigma_6 = \mu_3\phi_{10}.
\end{aligned}$$

This implies that $F = F_1 + \mu_3 F_2$, where F_1 was shown above to be simple and F_2 is given by

$$\begin{aligned}
F_2 = {} & e_{56789} - \eta_{10}e_{01567} + \eta_9 e_{01568} - \eta_8 e_{01569} - \eta_7 e_{01578} + \eta_6 e_{01579} - \eta_5 e_{01589}\\
& + \eta_4 e_{01678} - \eta_3 e_{01679} + \eta_2 e_{01689} - \eta_1 e_{01789} + \phi_{10} e_{02567} - \phi_9 e_{02568}\\
& + \phi_8 e_{02569} + \phi_7 e_{02578} - \phi_6 e_{02579} + \phi_5 e_{02589} - \phi_4 e_{02678} + \phi_3 e_{02679}\\
& - \phi_2 e_{02689} + \phi_1 e_{02789} + \tau_{10} e_{12567} - \tau_9 e_{12568} + \tau_8 e_{12569} + \tau_7 e_{12578} - \tau_6 e_{12579}\\
& + \tau_5 e_{12589} - \tau_4 e_{12678} + \tau_3 e_{12679} - \tau_2 e_{12689} + \tau_1 e_{12789} + \rho_5 e_{05678} - \rho_4 e_{05679}\\
& + \rho_3 e_{05689} - \rho_2 e_{05789} + \rho_1 e_{06789} + \sigma_5 e_{15678} - \sigma_4 e_{15679} + \sigma_3 e_{15689} - \sigma_2 e_{15789}\\
& + \sigma_1 e_{16789} - \lambda_5 e_{25678} + \lambda_4 e_{25679} - \lambda_3 e_{25689} + \lambda_2 e_{25789} - \lambda_1 e_{26789},
\end{aligned}$$

where the relations (28) hold and the independent parameters satisfy the same 10 equations (28). This then implies that

$$F_2 = \theta_5 \wedge \theta_6 \wedge \theta_7 \wedge \theta_8 \wedge \theta_9,$$

where

$$\theta_5 = e_5 + \rho_1 e_0 + \sigma_1 e_1 - \lambda_1 e_2, \qquad \theta_6 = e_6 + \rho_2 e_0 + \sigma_2 e_1 - \lambda_2 e_2,$$

$$\theta_7 = e_7 + \rho_3 e_0 + \sigma_3 e_1 - \lambda_3 e_2, \qquad \theta_8 = e_8 + \rho_4 e_0 + \sigma_4 e_1 - \lambda_4 e_2,$$

$$\theta_9 = e_9 + \rho_5 e_0 + \sigma_5 e_1 - \lambda_5 e_2.$$

Finally, we notice that the simple forms F_1 and F_2 are orthogonal since so are the 1-forms θ_i (defining $\theta_3 = e_3$ and $\theta_4 = e_4$). This then concludes the verification of the conjecture for this case.

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References

[1] M. Eastwood, P. Michor, Some remarks on the Plücker relations, Rendiconti del Circolo Matematico di Palermo, Serie II 63 (Suppl.) (2000) 85–88. math.AG/9905090.

[2] P. Griffiths, J. Harris, Principles of Algebraic Geometry, Wiley, New York, 1978.

[3] J. Figueroa-O'Farrill, Maximal supersymmetry in ten and eleven dimensions. math.DG/0109162.

[4] J. Figueroa-O'Farrill, G. Papadopoulos, Maximal supersymmetric solutions of ten- and eleven-dimensional supergravity. hep-th/0211089.

[5] M. Blau, J. Figueroa-O'Farrill, C. Hull, G. Papadopoulos, A new maximally supersymmetric background of type IIB superstring theory, J. High Energy Phys. 01 (2002) 047. hep-th/0110242.

[6] A. Chamseddine, J. Figueroa-O'Farrill, W. Sabra, Vacuum solutions of six-dimensional supergravities and Lorentzian Lie groups, in preparation.

[7] C. Boubel, Sur l'holonomie des variétés pseudo-Riemanniennes, Ph.D. Thesis, Université Henri Poincaré, Nancy I, 2000.

[8] V. Filippov, n-Lie algebras, Sibirsk. Mat. Zh. 26 (6) (1985) 126–140, 191.

[9] S. Salamon, Riemannian geometry and holonomy groups, Research Notes in Mathematics Series, vol. 201, Pitman, London, 1989.

[10] A. Medina, P. Revoy, Algébres de Lie et produit scalaire invariant, Ann. Sci. Éc. Norm. Sup. 18 (1985) 553.
[11] J. Figueroa-O'Farrill, S. Stanciu, Nonsemisimple Sugawara constructions, Phys. Lett. B 327 (1994) 40–46. hep-th/9402035.
[12] J. Figueroa-O'Farrill, S. Stanciu, On the structure of symmetric self-dual Lie algebras, J. Math. Phys. 37 (1996) 4121–4134. hep-th/9506152.